PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Law of Large Numbers

E. H. Linfoot

Phil. Trans. R. Soc. Lond. A 1928 **227**, 417-451
doi: 10.1098/rsta.1928.0011

XI. *On the Law of Large Numbers.*

By E. H. LINFOOT.

(*Communicated by* G. H. HARDY, *F.R.S.*)

(Received March 8, 1928.—Read June 7, 1928.)

1.1. *Introductory.*

Let p_i be an infinite sequence of numbers such that $0 \leq p_i \leq 1$ for $i = 1, 2, 3, \ldots \infty$. Suppose that n trials are made, the result of each trial being either a "hit" (T) or a "miss" (U); let the probability of a hit at the ith trial be p_i, of a miss q_i. Thus $p_i + q_i = 1$.

The number of hits in the n trials will be denoted by $m(n)$; the mathematical expectation of this number is of course $\sum_1^n p_i$. If then we write $m(n) = \mu(n) + \sum_1^n p_i$ we may call the number $\mu(n)$ so defined the "deviation." With $m(n)$ is associated a number t, defined by

$$m(n) = \sum_1^n p_i + t\sqrt{\sum_1^n 2p_i q_i}; \qquad \ldots \ldots \ldots \ldots (1)$$

evidently

$$t = \mu(n) \Big/ \sqrt{\sum_1^n 2p_i q_i}.$$

The probability that in the n trials there will be exactly m hits will be denoted by $P(m, n)$; the probability of at least m_0 hits will be written $P(m \geq m_0)$ or sometimes $P(t_0)$, where t_0 refers to (1).

The question we shall discuss is: What can be asserted about the order of $\mu(n)$? Certainly $\mu(n)$ may take all values from $-\sum_1^n p_i$ to $\sum_1^n q_i$, but it is well known, for instance, that as $n \to \infty$,

$$P\left(t_1\sqrt{\sum_1^n 2pq} \leq \mu(n) \leq t_2\sqrt{\sum_1^n 2pq}\right) \sim \frac{1}{\sqrt{\pi}}\int_{t_1}^{t_2} e^{-t^2}\,dt,$$

where $t_1 < t_2$ are constants and where $\sum_1^n 2pq \to \infty$ as $n \to \infty$ ("POISSON's formula").

In a previous paper I extended POISSON's formula to the case where t_1, t_2 are functions of n (not too rapidly increasing) and obtained a definite error term. In § 1 of the present paper analogous results are obtained for $P(t \geq t_0)$.

 [Published August 11, 1928.

These are then applied to the problem of finding an "exact upper band" for $|\mu(n)|$, *i.e.*, a function $f(n)$ such that we can assert with probability 1 first that $|\mu(n)| < (1+\varepsilon) f(n)$ for all large n and second that $|\mu(n)| > (1-\varepsilon) f(n)$ for an infinity of n. Here $\varepsilon > 0$ is arbitrarily small. This question was discussed by KHINTCHINE* in the case where all the probabilities are equal, and later† where they satisfy the wider condition

$$0 < a \leq p_i \leq 1 - a \qquad (i = 1, 2, \ldots \infty).$$

The main object of the present paper is the extension of his results to the most general class of cases:

$$\sum_1^n 2pq \to \infty \quad \text{as} \quad n \to \infty.$$

This extension occupies § 2.

§ 3 contains a general discussion of the inequality

$$|\mu(n)| < f(n),$$

from which it appears that KHINTCHINE's result can be improved upon. The concluding § 4 replaces his lower order-function for $|\mu(n)|$ by a sharper one.

Finally, I should like to thank Prof. BESICOVITCH for much valuable assistance and advice.

1.2. We begin by stating three results, established in the first paper, which will be required later. The first two are estimations of $P(m_1, m_2)$, one being in a more precise but less convenient form than the other:

$$P(m_1, m_2) = \frac{1}{\sqrt{\pi}} \int_{t_1 - \frac{1}{2}h}^{t_2 + \frac{1}{2}h} e^{-t^2} dt + \frac{1}{\sqrt{\pi}} \frac{\Sigma\, pq(p-q)}{3(\Sigma\, 2pq)^{3/2}} \int_{t_1 - \frac{1}{2}h}^{t_2 + \frac{1}{2}h} e^{-t^2} t(3 - 2t^2)\, dt$$

$$+ \theta_{19} [(\Sigma\, pq)^{-1+6\varepsilon} + \tfrac{9}{8} \sqrt{\Sigma\, 2pq}\, (t_2 - t_1 + 1)\, e^{-\frac{1}{2}(\Sigma\, pq)^{2\varepsilon}}], \quad . \quad . \quad . \quad \text{(A)}$$

where

$$h = 1/\sqrt{\Sigma 2pq} \quad \text{and} \quad |\theta_{19}| < 1,$$

provided

$$0 < \varepsilon < \tfrac{1}{6}, \quad (\Sigma\, pq)^{2\varepsilon} \geq 3, \quad \Sigma p + {}^{t_1}_{t_2} \sqrt{\Sigma\, 2pq}$$

are integers.

The second result asserts that for

$$|t_1|, \ |t_2| \leq \left(\sum_1^n 2pq\right)^{\frac{1}{6} - \varepsilon} \quad (0 < \varepsilon < \tfrac{1}{6}), \quad \eta > 0, \quad \text{and} \quad \sum_1^n 2pq \geq N(\eta, \varepsilon),$$

$$(1 - \eta) \frac{1}{\sqrt{\pi}} \int_{t_1}^{t_2} e^{-t^2} dt < P(t_1, t_2) < (1 + \eta) \frac{1}{\sqrt{\pi}} \int_{t_1}^{t_2} e^{-t^2} dt. \quad . \quad . \quad . \quad . \quad \text{(B)}$$

* 'Fund. Math.,' vol. 6, p. 9 (1924).

† 'Math. Ann.,' vol. 96, p. 152 (1925).

The third result is the estimation

$$\left|\prod_1^n (pe^{i\phi_0} + q)\right| = e^{0\cdot 051\,\theta_6\phi_0^4\Sigma pq - \frac{1}{2}\phi_0^2\Sigma pq} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(C)}$$

where $|\theta_6| < 1$.

1.3. *Discussion of* $P(m \geq m_0)$.—The next seven paragraphs (1.31–1.37) are based on an argument entirely different in type from that employed in the first paper, and lead to an inequality for $P(m \geq m_0)$ valid in a very general set of cases. We avoid trivial complications by assuming throughout that $\Sigma 2pq \geq 7$; this hypothesis is certainly satisfied for $n \geq n_0$ in our main investigation.

1.31. We have seen that the probability that exactly m hits will occur in n trials is the coefficient of x^m in the expansion of

$$\prod_1^n (p_i x + q_i)\,;$$

$P(m \geq m_0)$ is then the sum of the coefficients in this expansion from that of x^{m_0} onwards, and we can write

$$P(m \geq m_0) = \sum_{m \geq m_0} \cdots \sum p_{i_1} p_{i_2} \cdots p_{i_m} q_{i_{m+1}} \cdots q_{i_n}$$

(where $i_1, i_2, \ldots i_n$ are the numbers $1, 2, \ldots n$ in some order), or again

$$P_{m_0}(p_1, p_2, \ldots p_n) = \sum_{m \geq m_0} \cdots \sum p_{i_1} p_{i_2} \cdots p_{i_m} (1 - p_{i_{m+1}}) \ldots (1 - p_{i_n}). \quad . \quad (1.311)$$

Now suppose that the p's vary (n remaining fixed) so that

$$p_1 + p_2 + \ldots + p_n = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3121)$$

$$p_1^2 + p_2^2 + \ldots + p_n^2 = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3122)$$

Thus $\sum_1^n 2pq$ remains constant.

We shall investigate the circumstances in which P_{m_0} takes its greatest and least values. The polynomial $P_{m_0}(p_1, p_2, \ldots p_n)$ is a continuous function; these values are therefore attained. Consider the greatest value. A set of values $(p'_1, p'_2, \ldots p'_n)$ of the p's which gives P_{m_0} this value may be such that one or more of the p's are 0 or 1. We shall show that the remaining p's have one of two fixed values.

Suppose if possible that p'_1, p'_2, p'_3 are three of them, no two of which are equal. Let p_1, p_2, p_3 only vary, so that (1.3121) and (1.3122) are still maintained. Then

$$\left.\begin{aligned} p_1 + p_2 + p_3 &= p'_1 + p'_2 + p'_3 \\ p_1^2 + p_2^2 + p_3^2 &= p_1'^2 + p_2'^2 + p_3'^2 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3123)$$

and we are considering the variation of $P'_{m_0} = P_{m_0}(p_1, p_2, p_3, p'_4, p'_5 \dots p'_n)$. This is a symmetric function (not homogeneous) of degree three in p_1, p_2, p_3. If then

$$s_1 = p_1 + p_2 + p_3, \qquad s_2 = p_1^2 + p_2^2 + p_3^2, \qquad s_3 = p_1^3 + p_2^3 + p_3^3,$$

we can write

$$P'_{m_0} = f(s_1, s_2) + As_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3124)$$

where A is independent of p_1, p_2, p_3.

Now from (1.3123) we have

$$dp_1 + dp_2 + dp_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3131)$$

$$p_1\,dp_1 + p_2\,dp_2 + p_3\,dp_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3132)$$

At a maximum of P'_{m_0} we have also

$$dP'_{m_0} = 0.$$

From (1.3124) it follows that at the turning point (p'_1, p'_2, p'_3) we have

$$p_1^2\,dp_1 + p_2^2\,dp_2 + p_3^2\,dp_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3133)$$

(1.3131), (1.3132) and (1.3133) are thus satisfied simultaneously at (p'_1, p'_2, p'_3) by any values of dp_1, dp_2, dp_3 which satisfy the first two of them. They are therefore "consistent," and

$$\begin{vmatrix} 1 & 1 & 1 \\ p'_1 & p'_2 & p'_3 \\ p_1'^2 & p_2'^2 & p_3'^2 \end{vmatrix} = 0,$$

i.e.,

$$(p'_1 - p'_2)(p'_2 - p'_3)(p'_3 - p'_1) = 0.$$

This is a contradiction. Our assertion is therefore proved. The same argument applies to the least value of P_{m_0}, and we have consequently shown that:

"If the p's vary in accordance with (1.3121), (1.3122) the polynomial $P\,(m \geq m_0)$ assumes its greatest and likewise its least value when all the p's which are neither 0 nor 1 have one or other of two fixed values."

Such a set of p's, in which every one of $p_1, p_2, \dots p_n$ has one of the values $0, p_1^*, p_2^*, 1$, will be referred to as a "reduced set."

This is one of the most interesting results in the paper. We shall use it in (1.32) to obtain an estimation of $P\,(t \geq t_0)$. Using an argument similar to that of (1.32), we could establish (B) of (1.2) by a new method, reducing the discussion to the case where all the p's have one of two fixed values. We could even obtain a series estimation of the "error term" R on the right of

$$P\,(m_1, m_2) = \frac{1}{\sqrt{\pi}} \int_{t_1}^{t_2} e^{-t^2}\,dt + R$$

by supposing the p's altered in accordance with (1.3121) and (1.3122) so as to make $P(m_1, m_2)$ first a maximum, then a minimum, obtaining a series for R in these "reduced" cases, and using the fact that in the general case it lies between these extreme values. We shall not, however, go into this in the present paper.

1.32. The problem can thus be reduced to the case where all the p's have one or other of two fixed values. Suppose first that all the p's are equal, and consider the probability

$$P(t_0) = P(m \geq m_0) = \sum_{m \geq m_0} \frac{n!}{m!\; n-m!} p^m q^{n-m}. \quad . \quad . \quad . \quad . \quad (1.321)$$

The ratio of the $(r+1)$th term of this series to the $(r+2)$th is

$$\frac{n - m_0 - r}{m_0 + r + 1} \cdot \frac{p}{q} \qquad (0 \leq r \leq n - m_0 - 1).$$

This is always

$$\leq \frac{n - m_0}{m_0 + 1} \cdot \frac{p}{q} < 1, \quad \text{if} \quad t_0 > 0.$$

Thus

$$P(m \geq m_0) \leq \frac{n!}{m_0!\; n - m_0!} p^{m_0} q^{n-m_0} \left(1 + \left(\frac{n-m_0}{m_0+1}\right)\frac{p}{q} + \left(\frac{n-m_0}{m_0+1}\right)^2 \frac{p^2}{q^2} + \ldots \text{ to } \infty\right)$$

$$= P(m_0, n) \frac{(m_0+1)\, q}{q + t_0 \sqrt{2npq}} = P(m_0, n) \frac{q + npq + t_0 q \sqrt{2npq}}{q + t_0 \sqrt{2npq}}$$

$$= P(m_0, n)\left(q + \frac{(n+1)\, pq}{q + t_0 \sqrt{2npq}}\right) < P(m_0, n)\left(1 + \frac{\sqrt{2npq}}{t_0}\right). \quad . \quad . \quad (1.322)$$

Here t_0 is restricted by the condition that m_0 is to be an integer. But (1.322) is true without this restriction. Let m'_0 denote the least integer which is $\geq m_0$, m_0 now being any real number ≥ 0. Then $t'_0 \geq t_0$ and

$$P(t_0) = P(t'_0) < P(m'_0, n)(1 + \sqrt{2npq}/t'_0) < P(m'_0, n)(1 + \sqrt{2npq}/t_0).$$

If we agree to interpret $P(m, n)$ as $P(m', n)$ we have (since $P(t) = 0$ when $t' > nq/\sqrt{2npq}$),

$$P(t) < P(m, n)(1 + \sqrt{2npq}/t), \qquad \text{for all } t > 0. \quad . \quad . \quad . \quad . \quad (1.323)$$

1.33. Next let every p be either p_1^* or p_2^*. Suppose that $p = p_1^*$ in n_1 cases, $p = p_2^*$ in n_2 cases. Thus $n_1 + n_2 = n$. Let m_1, m_2 be the numbers of hits corresponding respectively to the n_1 trials where $p = p_1^*$, and to the n_2 trials where $p = p_2^*$, so that $m_1 + m_2 = m$. Thus

$$2n_1 p_1^* q_1^* + 2n_2 p_2^* q_2^* \to \infty \quad \text{as} \quad n \to \infty.$$

Consider

$$P(t_0) = P(m \geq n_1 p_1^* + n_2 p_2^* + t_0 \sqrt{2n_1 p_1^* q_1^* + 2n_2 p_2^* q_2^*}),$$

where $t_0 \geq 0$. If, adopting a slightly different notation from that used in previous sections, we write

$$t_1 \sqrt{2n_1p_1^*q_1^*} = t_2 \sqrt{2n_2p_2^*q_2^*} = \tfrac{1}{2}t_0 \sqrt{2n_1p_1^*q_1^* + 2n_2p_2^*q_2^*},$$

and observe that

$$m_1 + m_2 \geq n_1p_1^* + n_2p_2^* + t_0\sqrt{2n_1p_1^*q_1^* + 2n_2p_2^*q_2^*}$$

implies one at least of

$$m_1 \geq n_1p_1^* + t_1 \sqrt{2n_1p_1^*q_1^*},$$

$$m_2 \geq n_2p_2^* + t_2 \sqrt{2n_2p_2^*q_2^*},$$

we see that

$$\mathrm{P}(t) \leq \mathrm{P}(m_1 \geq n_1p_1^* + t_1 \sqrt{2n_1p_1^*q_1^*}) + \mathrm{P}(m_2 \geq n_2p_2^* + t_2 \sqrt{2n_2p_2^*q_2^*})$$
$$= \mathrm{P}_1 + \mathrm{P}_2, \text{ say.} \quad . \quad . \quad . \quad (1.331)$$

1.34. Now it is well known (see *e.g.*, CHRYSTAL, 'Algebra II,' p. 368) that for $n \geq 2$,

$$n\,! = \sqrt{2\pi n}\,(n/e)^n\, e^{\frac{1}{12n}+\theta}, \quad \text{where} \quad -1/24n^2 < \theta < 1/24n\,(n-1),$$

and we observe that the inequality still holds for $n = 1$.

Thus if $1 \leq m \leq n - 1$,

$$\mathrm{P}(m, n) = \frac{n\,!}{m\,!\;\, n-m\,!}\, p^m\, q^{n-m}$$

$$< \sqrt{\frac{n}{2\pi m\,(n-m)}}\, \frac{n^n\, p^m\, q^{n-m}}{m^m\,(n-m)^{n-m}} \exp\left(\frac{1}{12n} + \frac{1}{24n\,.\,n-1} - \frac{1}{12m} + \frac{1}{24m^2} - \frac{1}{12\,(n-m)} + \frac{1}{24\,(n-m)^2}\right)$$

$$\leq \sqrt{\frac{n}{2\pi\,(n-1)}} \left(\frac{np}{m}\right)^m \left(\frac{nq}{n-m}\right)^{n-m} e^{1/6}.$$

Writing

$$\mathrm{A}_m{}^n = (np/m)^m\,(nq/n-m)^{n-m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.342)$$

we have, for $n \geq 2$, $1 \leq m \leq n - 1$ in the first place,

$$\mathrm{P}(m, n) \leq \mathrm{A}_m{}^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.343)$$

Let $\mathrm{A}_m{}^n$ be interpreted as q^n when $m = 0$, p^n when $m = n$, so that $\mathrm{A}_m{}^n$ is a continuous function of the real variable m in the interval $0 \leq m \leq n$. Then since

$$\mathrm{P}(0, n) = q^n, \qquad \mathrm{P}(n, n) = p^n,$$

we see that (1.343) holds provided only that the integers n, m satisfy

$$n \geq 1, \qquad 0 \leq m \leq n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3431)$$

Let us now introduce the additional hypothesis that $m/n \geq p$.

Then writing $m/n = \bar{p}_m$, $1 - \bar{p}_m = \bar{q}_m$, we have $A_m = (p/\bar{p}_m)^{\bar{p}_m} (q/\bar{q}_m)^{\bar{q}_m}$ with

$$0 < p < 1, \qquad p \leq \bar{p}_m \leq 1.$$

Whence

$$\begin{aligned}\log A_m &= \bar{p}_m \log p + (1 - \bar{p}_m) \log (1-p) - \bar{p}_m \log \bar{p}_m - (1-p) \log (1 - \bar{p}_m) \\ &= (\bar{p}_m - p) \log p + [(1 - \bar{p}_m) - (1-p)] \log (1-p) \\ &\qquad + [p \log p + (1-p) \log (1-p)] - [\bar{p}_m \log \bar{p}_m + (1 - \bar{p}_m) \log (1 - \bar{p}_m)] \\ &= -\tfrac{1}{2} (\bar{p}_m - p)^2 (1/p' + 1/(1-p'))\end{aligned}$$

where p' is a point of $(p, \bar{p}_m)$, interior unless $p = \bar{p}_m$.

Thus

$$A_m^n = \exp \{-\tfrac{1}{2} n (\bar{p}_m - p)^2 (1/p + 1/(1-p'))\} \quad . \quad . \quad . \quad . \quad (1.344)$$

for $n \geq 1$, $pn \leq m \leq n$, where p' is some number in the interval $(p, \bar{p}_m)$.

1.35. We can apply this inequality to find an upper bound to P_1, P_2 of (1.33) on the assumption that $t_0 \geq 0$. For reasons which will appear later we suppose that

$$0 \leq t_0 \leq \sqrt{2n_1 p_1^* q_1^* + 2n_2 p_2^* q_2^*}.$$

Consider first

$$P_1 = P_1 (m_1 \geq n_1 p_1^* + \tfrac{1}{2} t_0 \sqrt{\Sigma 2pq}).$$

Let $m'_1 = n_1 p_1^* + t'_1 \sqrt{2n_1 p_1^* q_1^*}$ be the least integer $\geq n_1 p_1^* + \frac{1}{2} t_0 \sqrt{\Sigma 2pq}$. Then by (1.343), (1.3431), (1.344)

$$P(m'_1, n_1) < \exp \{-\tfrac{1}{2} n_1 (\bar{p}_{m'_1} - p_1^*)^2 (1/p' + 1/(1-p'))\}$$

for $n_1 \geq 1$, $0 \leq m'_1 \leq n$, where p' lies in $(p_1^*, \bar{p}_{m'_1})$.

There are several cases to consider :

(1) $\bar{p}_{m'_1} > p_1^* \geq \frac{1}{2}$. Then

$$\frac{1}{p'} + \frac{1}{1-p'} > \frac{1}{p_1^*} + \frac{1}{1-p_1^*} = \frac{1}{p_1^* q_1^*},$$

and

$$P(m'_1, n) < \exp\left\{-\frac{n_1}{2p_1^* q_1^*} \cdot \frac{t_0^2}{4n_1^2} (2n_1 p_1^* q_1^* + 2n_2 p_2^* q_2^*)\right\} \leq e^{-\frac{t_0^2}{4}}. \quad . \quad (1.351)$$

(2) $\frac{1}{2} \geq \bar{p}_{m'_1} > p_1^*$. Then

$$\frac{1}{p'} + \frac{1}{1-p'} > \frac{1}{\bar{p}_{m'_1}} + \frac{1}{1-\bar{p}_{m'_1}} = \frac{1}{\bar{p}_{m'_1} \bar{q}_{m'_1}},$$

and (since $t'_1 \sqrt{2n_1p_1^*q_1^*} \geq t_1 \sqrt{2n_1p_1^*q_1^*} = \frac{1}{2}t_0 \sqrt{2n_1p_1^*q_1^* + 2n_2p_2^*q_2^*} \geq 0$)

$$\mathrm{P}(m'_1, n_1) \leq \exp\left\{-n_1 \cdot \frac{t'^2_1 \, . \, 2n_1p_1^*q_1^*}{2\,(n_1p_1^* + t'_1\sqrt{2n_1p_1^*q_1^*})\,(n_1q_1^* - t'_1\sqrt{2n_1p_1^*q_1^*})}\right\}$$

$$\leq \exp\left\{-\frac{t'^2_1 \, . \, 2n_1p_1^*q_1^*}{2n_1p_1^*q_1^* + 2\,(q_1^* - p_1^*)\,t'_1\sqrt{2n_1p_1^*q_1^*}}\right\}$$

$$\leq \exp\left\{-\frac{t'^2_1 \, . \, 2n_1p_1^*q_1^*}{2n_1p_1^*q_1^* + 2t'_1\sqrt{2n_1p_1^*q_1^*}}\right\}$$

$$\leq \exp\left\{-\frac{t_1^2\, 2n_1p_1^*q_1^*}{2np_1^*q_1^* + 2t_1\sqrt{2n_1p_1^*q_1^*}}\right\}$$

$$= \exp\left\{-\tfrac{1}{4}t_0^2 \frac{2n_1p_1^*q_1^* + 2n_2p_2^*q_2^*}{2n_1p_1^*q_1^* + t_0\sqrt{\Sigma\, 2pq}}\right\}.$$

If $t_0\sqrt{\Sigma\, 2pq} \leq 2n_1p_1^*q_1^*$, this $\leq \exp\left(-\frac{1}{8}t_0^2 \, . \, \frac{2n_1p_1^*q_1^* + 2n_2p_2^*q_2^*}{2n_1p_1^*q_1^*}\right) \leq \exp(-t_0^2/8)$.

If $t_0\sqrt{\Sigma\, 2pq} > 2n_1p_1^*q_1^*$, it is less than $\exp(-\frac{1}{8}t_0\sqrt{\Sigma\, 2pq}) \leq \exp(-t_0^2/8)$.

Thus in either case

$$\mathrm{P}(m'_1, n_1) \leq \exp(-t_0^2/8). \quad \quad (1.352)$$

(3) $\bar{p}_{m'_1} \geq \frac{1}{2} > p_1 > \frac{1}{16}$. Then since

$$\frac{1}{p'} + \frac{1}{1-p'} \geq 4,$$

$$\mathrm{P}(m'_1, n_1) \leq \exp(-4t'^2_1p_1^*q_1^*) \leq \exp(-\tfrac{15}{64}t'^2_1) \leq \exp(-\tfrac{15}{64}t_1^2)$$

$$\leq \exp(-\tfrac{15}{256}t_0^2) \leq \exp(-\tfrac{1}{18}t_0^2). \quad \quad (1.353)$$

(4) $p_1 \leq \frac{1}{16}$, $\bar{p}_{m'_1} \geq \frac{1}{2}$. Then since

$$t_0 \leq \sqrt{\Sigma\, 2pq}, \qquad m'_1 \geq \tfrac{1}{2}n_1,$$

$$n_1(\tfrac{1}{2} - p_1) < \tfrac{1}{2}(2n_1p_1^*q_1^* + 2n_2p_2^*q_2^*) + 1,$$

$$n_1 \leq \tfrac{8}{7}(\Sigma 2pq + 2) < \tfrac{3}{2}\Sigma 2pq \quad \text{if} \quad \Sigma 2pq \geq 7.$$

Thus

$$t_1\sqrt{2n_1p_1^*q_1^*} = \tfrac{1}{2}t_0\sqrt{\Sigma 2pq} \geq \tfrac{1}{2}t_0\sqrt{\tfrac{2}{3}}\,n_1^{1/2}.$$

If now $t_0 > \sqrt{6n_1}$ this gives, since $t'_1 \geq t_1$,

$$t'_1\sqrt{2n_1p_1^*q_1^*} > n_1,$$

and *a fortiori*

$$m'_1 > n_1,$$

which is impossible. It follows that in the case we are discussing,

$$t_0 \leq \sqrt{6n_1}. \qquad (1.3541)$$

Now increasing m by 1 multiplies $A_m{}^n$ by

$$\frac{np}{nq} \cdot \frac{m^m (n-m)^{n-m}}{(m+1)^{m+1}(n-m-1)^{n-m-1}}. \qquad (1.3542)$$

Consider $\phi(t) = t^t (a-t)^{a-t}$ for $a/2 \leq t \leq a$. ($\phi(a)$ is defined as $\lim_{t \to a-0} \phi(t)$). It is an increasing function. For

$$D \log \phi(t) = \log t - \log(a-t) \geq 0.$$

It follows that for $m \geq \frac{1}{2}n$, $p < \frac{1}{2}$ the expression (1.3542) is less than 1.

Thus $A_{m_1}^{n_1}$ is greatest when m_1 has for its value the least integer $\geq \frac{1}{2}n_1$. For $n_1 \geq 2$ we may therefore assume $\bar{p}_{m'_1} \leq \frac{2}{3}$. Then

$$A_{m'_1} = \left(\frac{p_1{}^*}{\bar{p}_{m'_1}}\right)^{\bar{p}_{m'_1}} \left(\frac{q_1{}^*}{\bar{q}_{m'_1}}\right)^{\bar{q}_{m'_1}} \leq \left(\frac{p_1{}^* q_1{}^*}{\bar{p}_{m'_1}\,\bar{q}_{m'_1}}\right)^{\bar{q}_{m'_1}} \leq \left(\frac{\frac{1}{16} \cdot \frac{15}{16}}{\frac{2}{3} \cdot \frac{1}{3}}\right)^{1/3} = \left(\frac{135}{512}\right)^{1/3}$$

$$P(m'_1, n_1) \leq A_{m_1}^{n_1} \leq \exp(-\tfrac{1}{3}n_1 \log 512/135) < \exp(-\tfrac{1}{3}n_1)$$
$$\leq \exp(-t_0{}^2/18) \text{ by } (1.3541).$$

This holds for $n_1 \geq 2$, $\Sigma 2pq \geq 7$. When $n_1 = 1$, $\bar{p}_{m'_1} \geq \frac{1}{2}$ gives $m'_1 = 1$. As before $t_0 \leq \sqrt{6n_1} = \sqrt{6}$ if $\Sigma 2pq \geq 7$.

Therefore

$$P(m'_1, n_1) = p_1{}^* < e^{-\log 2} < e^{-1/2} < e^{-t_0{}^2/12}. \qquad (1.354)$$

Thus

"For $n_1 \geq 1$, $0 \leq t_0 \leq \sqrt{2n_1 p_1{}^* q_1{}^* + 2n_2 p_2{}^* q_2{}^*}$, $2n_1 p_1{}^* q_1{}^* + 2n_2 p_2{}^* q_2{}^* \geq 7$,

$$P(m'_1, n_1) < \exp(-t_0{}^2/18)\text{"} \qquad (1.355)$$

The same inequality holds for $P(m'_2, n_2)$, when $n_2 \geq 1$.

1.36. Using (1.323) we now have

$$P_1 < \exp(-t_0{}^2/18) \, . \, (1 + \sqrt{2n_1 p_1{}^* q_1{}^*}/t_1) \quad \text{for} \quad 0 < t_0 \leq \sqrt{\Sigma 2pq},\ \Sigma 2pq \geq 7,\ n_1 \geq 1.$$
$$P_2 < \exp(-t_0{}^2/18) \, . \, (1 + \sqrt{2n_2 p_2{}^* q_2{}^*}/t_2) \quad \text{for} \quad 0 < t_0 \leq \sqrt{\Sigma 2pq},\ \Sigma 2pq \geq 7,\ n_2 \geq 1.$$

Observing that (1.331) is true in the case $n_1 = 0$ if we then interpret P_1 as zero we conclude that

$$P(t_0) < \exp(-t_0{}^2/18) \, . \, (2 + \sqrt{2n_1 p_1 q_1}/t_1 + \sqrt{2n_2 p_2 q_2}/t_2)$$
$$= 2 \exp(-t_0{}^2/18) \, . \, (1 + \sqrt{\Sigma 2pq}/t_0) \qquad (1.361)$$

for $0 < t_0 \leq \sqrt{\Sigma 2pq}$, $\Sigma 2pq \geq 7$.

Finally we have only to consider misses instead of hits, interchange the rôles of p and q, and observe that if $m = np + t\sqrt{\Sigma 2pq}$, then $n - m = nq - t\sqrt{\Sigma 2pq}$, to obtain the analogous result

$$P'(t_0) = P(m \leq np - t_0\sqrt{\Sigma 2pq}) < 2\exp(-t_0^2/18)(1 + \sqrt{\Sigma 2pq}/t_0)$$

for

$$0 < t_0 \leq \sqrt{\Sigma 2pq}, \quad \Sigma 2pq \geq 7. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1.362)$$

1.37. Now let the p's be unrestricted, even by the condition $\sum_1^n 2pq \to \infty$ as $n \to \infty$. The expression for $P(m \geq m_0)$ is a polynomial in $p_1, p_2, \ldots p_n$, and we saw in (1.31) that it takes its greatest value $\bar{P}(m \geq m_0)$ when the p's all have one or other of the values, 0, $p_1{}^*$, $p_2{}^*$, 1. The conditions restricting the variation of the p's may be stated in the form

$$p_1 + p_2 + \ldots + p_n = \text{const.}$$
$$2p_1q_1 + 2p_2q_2 + \ldots + 2p_nq_n = \text{const.}$$

Let s be the number of p's which are 1 in the reduced set, n_1 the number which are $p_1{}^*$, n_2 the number which are $p_2{}^*$. As in (1.33) let m_1 be the number of hits corresponding to the n_1 trials where $p = p_1{}^*$, m_2 the number corresponding to the trials where $p = p_2{}^*$. Then

$$\bar{P}(m \geq m_0) = P(m \geq n_1p_1{}^* + n_2p_2{}^* + s + t_0\sqrt{2n_1p_1{}^*q_1{}^* + 2n_2p_2{}^*q_2{}^*})$$
$$= P(m_1 + m_2 \geq n_1p_1{}^* + n_2p_2{}^* + t_0\sqrt{2n_1p_1{}^*q_1{}^* + 2n_2p_2{}^*q_2{}^*}).$$

For if we remove from the reduced set a trial at which $p_\kappa = 0$ and recalculate the probability for at least m_0 hits the new polynomial is the same as the old. On the other hand, if we remove a trial at which $p_\kappa = 1$ it is the polynomial which gives the probability for at least $m_0 - 1$ hits in the new set which now coincides with the old one for at least m_0 hits. Thus the above equality is proved by induction. (To say that the probability of an event is zero is not, of course, to deny its possibility.)

We infer from (1.361), (1.362) that:

"For all values of the probabilities p_κ which make $\sum_1^n 2p_\kappa q_\kappa \geq 7$, and for all real t_0, such that $0 < t_0 \leq \sqrt{\sum_1^n 2pq}$,

$$P\left(m \geq \sum_1^n p_\kappa + t_0\sqrt{\sum_1^n 2p_\kappa q_\kappa}\right) < 2\exp(-t_0^2/18)(1 + \sqrt{\Sigma 2pq}/t_0), \quad . \qquad (1.371)$$

and

$$P\left(m \leq \sum_1^n p_\kappa - t_0\sqrt{\sum_1^n 2p_\kappa q_\kappa}\right) < 2\exp(-t_0^2/18)(1 + \sqrt{\Sigma 2pq}/t_0)." \qquad (1.372)$$

1.38. From these inequalities and (B) we can now deduce more elegant results, valid for all sufficiently large values of $\sum_1^n 2pq$ and for $t_0(n) = O((\Sigma 2pq)^{1/6-\epsilon})$.

Lemma 1.381.—If $t_0 \geq \frac{1}{2}$, then

$$\int_{t_0}^{\infty} e^{-n^2}\,dn > \frac{1}{8t_0}\,e^{-t_0^2}.$$

Proof.

$$\int_{t_0}^{\infty} e^{-n^2}\,dn > \int_{t_0}^{t_0+1/4t_0} e^{-n^2}\,dn > \frac{1}{4t_0}\exp\left(-t_0^2 - \tfrac{1}{2} - \frac{1}{16t_0^2}\right) > \frac{1}{8t_0}\,e^{-t_0^2}.$$

Suppose

$$|\,t_1\,| = \left|\,t_1\left(\sum_1^n 2pq\right)\right| \leq (\Sigma 2pq)^{1/6-\epsilon}. \qquad (0 < \epsilon < \tfrac{1}{6}.)$$

Let

$$t_2 = t_2\left(\sum_1^n 2pq\right) = (\Sigma 2pq)^{1/6-\epsilon/2}.$$

Then the hypotheses of (B) are satisfied by t_1, t_2 and

$$\mathrm{P}(t_1, t_2) = \mathrm{P}(m_1 \leq m \leq m_2)$$
$$< (1 + \tfrac{1}{2}\eta)\frac{1}{\sqrt{\pi}}\int_{t_1}^{t_2} e^{-t^2}\,dt \text{ for all large } \sum_1^n 2pq.$$

Whence

$$\mathrm{P}(t_1) = \mathrm{P}(t_1, t_2) + \mathrm{P}(t_2) - \mathrm{P}(m = m_2)$$
$$\leq \mathrm{P}(t_1, t_2) + \mathrm{P}(t_2)$$
$$< (1 + \tfrac{1}{2}\eta)\frac{1}{\sqrt{\pi}}\int_{t_1}^{t_2} e^{-t^2}\,dt + 2\exp\{-\tfrac{1}{18}(\Sigma 2pq)^{1/3-\epsilon}\}\,.\,(1 + (\Sigma 2pq)^{5/2+\epsilon/2})$$

for $\sum_1^n 2pq \geq \mathrm{N}_1(\eta, \epsilon)$. Now

$$2\exp\{-\tfrac{1}{18}(\Sigma 2pq)^{1/3-\epsilon}\}\,.\,(1 + (\Sigma 2pq)^{5/2+\epsilon/2}) < \frac{\eta}{2\sqrt{\pi}}\,\frac{\exp\{-(\Sigma 2pq)^{1/3-2\epsilon}\}}{8\,(\Sigma 2pq)^{1/6-\epsilon}}$$
$$\text{for } \Sigma 2pq \geq \mathrm{N}_2(\eta, \epsilon)$$
$$< \tfrac{1}{2}\eta\,\frac{1}{\sqrt{\pi}}\int_{t_1}^{\infty} e^{-t^2}\,dt.$$

Thus

$$\mathrm{P}(t_1) < (1 + \eta)\frac{1}{\sqrt{\pi}}\int_{t_1}^{\infty} e^{-t^2}\,dt$$
$$\text{for } \sum_1^n 2pq \geq \mathrm{N}_3(\eta, \epsilon), \qquad \left|\,t_1\left(\sum_1^n 2pq\right)\right| \leq \left(\sum_1^n 2pq\right)^{1/6-\epsilon}.$$

Also

$$\mathrm{P}(t_1) > \mathrm{P}(t_1, t_2)$$
$$> (1 - \tfrac{1}{2}\eta)\frac{1}{\sqrt{\pi}}\int_{t_1}^{(\Sigma 2pq)^{1/6-\epsilon/2}} e^{-t^2}\,dt, \quad \text{for} \quad \sum_1^n 2pq \geq \mathrm{N}_4(\eta, \epsilon)$$
$$> (1 - \eta)\frac{1}{\sqrt{\pi}}\int_{t_1}^{\infty} e^{-t^2}\,dt, \quad \text{for} \quad \sum_1^n 2pq \geq \mathrm{N}_5(\eta, \epsilon).$$

Thus :

"If $t = t\left(\sum_1^n 2pq\right)$, $\varepsilon > 0$ $\eta > 0$ and $|t| \leq (\Sigma 2pq)^{1/6-\varepsilon}$

then

$$(1-\eta)\frac{1}{\sqrt{\pi}}\int_t^\infty e^{-t^2}\,dt < \mathrm{P}\,(m \geq \Sigma p + t\sqrt{\Sigma 2pq}) < (1+\eta)\frac{1}{\sqrt{\pi}}\int_t^\infty e^{-t^2}\,dt \quad . \quad . \quad (1.382)$$

for all $\sum_1^n 2pq \geq \mathrm{N}_0(\eta, \varepsilon)$."

A sharper estimation can evidently be obtained if we narrow the limits for t. Taking $\varepsilon = \frac{1}{12}$, for instance, we easily obtain

$$\mathrm{P}\,(m \geq \Sigma p + t\sqrt{\Sigma 2pq}) = \frac{1}{\sqrt{\pi}}\int_t^\infty e^{-t^2}\,dt + \theta \exp\{-(\Sigma 2pq)^{1/4}\} \quad (\text{where } |\theta| < 1) \quad . \quad . \quad . \quad . \quad (1.383)$$

for $|t| \leq \left(\sum_1^n 2pq\right)^{1/12}$ and all large $\sum_1^n 2pq$.

The same results hold for $\mathrm{P}\,(m \leq \Sigma p - t\sqrt{\Sigma 2pq})$ of course, the integral $\int_{-t}^\infty e^{-t^2}\,dt$ being replaced by $\int_{-\infty}^t e^{-t^2}\,dt$.

All these inequalities have been stated in terms of $\sum_1^n 2pq$ only, and hold uniformly in the individual p's and q's. We have only to choose a particular sequence $p_1, p_2, p_3, \ldots$, such that $\sum_1^n 2pq \to \infty$ with n to obtain as a special case of (1.382) :

"If $p_1, p_2, p_3, \ldots$ are such that $\sum_1^n 2pq \to \infty$ as $n \to \infty$, and if $t = t(n)$ is such that

$$|t| \leq \left(\sum_1^n 2pq\right)^{1/6-\varepsilon} \qquad (\text{where } \varepsilon > 0),$$

Then given $\eta > 0$ we have

$$(1-\eta)\frac{1}{\sqrt{\pi}}\int_t^\infty e^{-t^2}\,dt < \mathrm{P}\left(m \geq \sum_1^n p + t\sqrt{\sum_1^n 2pq}\right) < (1+\eta)\frac{1}{\sqrt{\pi}}\int_t^\infty e^{-t^2}\,dt$$

for all $n \geq n_0(\eta, \varepsilon)$."

1.4. We conclude this first section by stating explicitly a trivial corollary of (1.382) :

"If $t = t\left(\sum_1^n 2pq\right) \leq \left(\sum_1^n 2pq\right)^{1/6-\varepsilon}$ $(\varepsilon > 0)$ and $t \to \infty$ as $\sum_1^n 2pq \to \infty$, then

$$\mathrm{P}\left(|\mu(n)| \geq t\sqrt{\sum_1^n 2pq}\right) < \tfrac{1}{2}e^{-t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.41)$$

for all sufficiently large values of $\sum_1^n 2pq$."

Proof.

$$\int_t^\infty e^{-n^2}\,dn < \frac{e^{-t^2}}{2t} \text{ for } t \geq \tfrac{1}{2}.$$

2.1. The next section is a generalisation of a paper of KHINTCHINE'S ('Math. Annalen,' vol. 96, p. 153). The problem he solves is that of finding "an exact upper bound" to the deviation

$$\mu(n) = m(n) - \sum_1^n p_i.$$

Definition.—By an exact upper bound of the function $\mu(n)$ is meant a positive function $\chi(n)$ which satisfies the following condition:

"Given $\delta > 0$, $\eta > 0$ we have with probability $> 1 - \eta$

(1) $|\mu(n)|/\chi(n) < 1 + \delta$ for $n \geq n_0(\delta, \eta)$,

(2) $|\mu(n)|/\chi(n) > 1 - \delta$ for an infinite sequence of values of the integer n." . . . . (2.11)

The condition satisfied by $\chi(n)$ may be stated slightly more formally as follows:

"Given $\delta > 0$, $\eta > 0$ we can find an arbitrarily large positive integer $n_0 = n_0(\delta, \eta)$ such that we can make, with probability $> 1 - \eta$, the double assertion

(1) $|\mu(n)|/\chi(n) < 1 + \delta$ for all $n \geq n_0$,

(2) $|\mu(n)|/\chi(n) > 1 - \delta$ for at least one $n \geq n_0$." . . . . (2.12)

It is plain that the assertion "$\chi(n)$ is an exact upper bound to $\mu(n)$" defines a class of functions $\chi(n)$; our problem is to find a member of this class, and it is not evident *a priori* that a member exists which can be represented by an analytical expression.

KHINTCHINE showed that if p_i, $q_i \geq a > 0$ for $i = 1, 2, \ldots$, then

$$\chi(n) = \sqrt{\sum_1^n 2p_iq_i \log\log n}$$

is a solution to the problem. He observed that this was asymptotic to

$$\sqrt{\sum_1^n 2p_iq_i \log\log \sum_1^n 2p_iq_i}$$

(which was therefore equally a solution), and suggested that this last expression held in a more general class of cases.

We prove the truth of his supposition for the (most general) class defined by

$$0 \leq p, q \leq 1, \qquad \sum_1^n 2pq \to \infty \text{ with } n.$$

Our argument, like KHINTCHINE'S, is a generalisation of one due to HARDY and LITTLEWOOD.* They considered the expansion of an arbitrary real number of the interval (0, 1) as a decimal in the scale of the arbitrary integer a, and showed that the deviation

* 'Acta Math.,' vol. 37, p. 155 (p. 183) (1914).

$\mu(n)$ of the number (say) of zeros occurring in the first n places from the "correct" number n/a almost always* satisfied the double relation

(1) $|\mu(n)|/\sqrt{n \log n}$ is bounded for all large n,

(2) $|\mu(n)|/\sqrt{n} > c$ for an infinity of n,

$c > 0$ being chosen arbitrarily.

Using the arguments of § 1, we can obtain the solution of the generalised problem which corresponds to the "Poisson" series of divisions.

2.21. Let s_x be the least positive integer such that

$$\sum_1^{s_x} 2p_i q_i \geq x,$$

and for $n \geq s_e = s_{2\cdot71828\ldots}$ let $\chi(n)$ always denote $\sqrt{\sum_1^n 2pq \log\log \sum_1^n 2pq}$.

2.22. *Lemma.*

If $B(\kappa_1, \kappa_2)$ denote the probability of the relation

$$|\mu(\kappa_1)/\chi(\kappa_1) - \mu(\kappa_2)/\chi(\kappa_2)| > \varepsilon \qquad (0 < \varepsilon < 1);$$

then for $n_1 + n_1^{11/12} < n_2 < 2n_1$

$$B(s_{n_1}, s_{n_2}) < \exp\left(-\frac{\varepsilon^2}{4}\frac{n_1 \operatorname{ll} n_1}{n_2 - n_1}\right), \quad \ldots \ldots \ldots \quad (2.22)$$

where ll stands for log log.

Proof.—Let the event T appear $m(\kappa_1, \kappa_2)$ times in the succession of trials from the (κ_1+1)th to the κ_2th, and let

$$\mu(\kappa_1, \kappa_2) = m(\kappa_1, \kappa_2) - \sum_{\kappa_1+1}^{\kappa_2} p_i,$$

so that

$$\mu(\kappa_2) = \mu(\kappa_1) + \mu(\kappa_1, \kappa_2).$$

Then

$$B(s_{n_1}, s_{n_2}) = P(|\mu(s_{n_1})/\chi(s_{n_1}) - \mu(s_{n_2})/\chi(s_{n_2})| > \varepsilon)$$
$$= P(|\mu(s_{n_1})\{1/\chi(s_{n_1}) - 1/\chi(s_{n_2})\} - \mu(s_{n_1}, s_{n_2})/\chi(s_{n_2})| > \varepsilon).$$

Now

$$|\mu(s_{n_1})\{1/\chi(s_{n_1}) - 1/\chi(s_{n_2})\} - \mu(s_{n_1}, s_{n_2})/\chi(s_{n_2})| > \varepsilon \quad \ldots \quad (2.221)$$

involves one at least of

$$|\mu(s_{n_1})\{1/\chi(s_{n_1}) - 1/\chi(s_{n_2})\}| > \tfrac{1}{2}\varepsilon, \qquad |\mu(s_{n_1}, s_{n_2})/\chi(s_{n_2})| > \tfrac{1}{2}\varepsilon. \quad . \quad (2.222, 2.223)$$

Therefore

$$B(s_{n_1}, s_{n_2}) \leq B_1(s_{n_1}, s_{n_2}) + B_2(s_{n_1}, s_{n_2}), \quad \ldots \ldots \quad (2.224)$$

where

$$B_1(s_{n_1}, s_{n_2}) = P(|\mu(s_{n_1})\{1/\chi(s_{n_1}) - 1/\chi(s_{n_2})\}| > \tfrac{1}{2}\varepsilon,$$
$$B_2(s_{n_1}, s_{n_2}) = P(|\mu(s_{n_1}, s_{n_2})/\{\chi(s_{n_2})\}| > \tfrac{1}{2}\varepsilon.$$

* *I.e.* for all real numbers except a set of measure zero.

When (2.222) is satisfied we have

$$|\mu(s_{n_1})| > \tfrac{1}{2}\varepsilon\,\chi(s_{n_1})\,\chi(s_{n_2})/\{\chi(s_{n_2}) - \chi(s_{n_1})\}$$

$$> \tfrac{1}{2}\varepsilon\,\sqrt{n_1 \,\mathrm{ll}\, n_1}\,\sqrt{n_2\,\mathrm{ll}\,n_2}/\{\sqrt{(n_2+1)\,\mathrm{ll}\,(n_2+1)} - \sqrt{n_1\,\mathrm{ll}\,n_1}\}.$$

Now if

$$\phi(t) = \sqrt{t\,\mathrm{ll}\,t}, \qquad \mathrm{D}\phi(t) = \frac{\sqrt{\mathrm{ll}\,t}}{2\sqrt{t}}\left(1 + \frac{1}{\mathrm{l}\,t\,\mathrm{ll}\,t}\right),$$

and this is a decreasing function when t is large. Thus

$$\sqrt{(n_2+1)\,\mathrm{ll}\,(n_2+1)} - \sqrt{n_1\,\mathrm{ll}\,n_1} = (n_2 - n_1)\frac{\sqrt{\mathrm{ll}\,n'}}{2\sqrt{n'}}\left(1 + \frac{1}{\mathrm{l}\,n'\,\mathrm{ll}\,n'}\right), \quad \text{where } n_1 < n' < n_2.$$

$$< (n_2 - n_1)\frac{\sqrt{\mathrm{ll}\,n_1}}{2\sqrt{n_1}}\left(1 + \frac{1}{\mathrm{l}\,n_1\,\mathrm{ll}\,n_1}\right), \quad \text{for all large } n_1,$$

$$< \sqrt{2}\,(n_2 - n_1)\frac{\sqrt{\mathrm{ll}\,n_2}}{2\sqrt{n_1}} \qquad (n_1 \text{ large}).$$

Again,

$$2(\sqrt{n_2\,\mathrm{ll}\,n_2} - \sqrt{n_1\,\mathrm{ll}\,n_2}) = 2\sqrt{\mathrm{ll}\,n_2}\,(\sqrt{n_2} - \sqrt{n_1}) = 2\sqrt{\mathrm{ll}\,n_2}\,(n_2 - n_1)/2\sqrt{n''},$$

$$(\text{where } n_1 < n'' < n_2)$$

$$> \sqrt{2}\,(n_2 - n_1)\,\sqrt{\mathrm{ll}\,n_2}/2\sqrt{n_1}.$$

It follows that, for $n_1 > \mathrm{N}_0$,

$$\sqrt{(n_2+1)\,\mathrm{ll}\,(n_2+1)} - \sqrt{n_1\,\mathrm{ll}\,n_1} < 2(\sqrt{n_2} - \sqrt{n_1})\,\sqrt{\mathrm{ll}\,n_2},$$

and we therefore have as a consequence of (2.222)

$$|\mu(s_{n_1})| > \tfrac{1}{4}\varepsilon\,\sqrt{n_1\,\mathrm{ll}\,n_1}\,\sqrt{n_2}/(\sqrt{n_2} - \sqrt{n_1}) = \tfrac{1}{4}\varepsilon\,\sqrt{n_1\,\mathrm{ll}\,n_1}\,(\sqrt{n_1 n_2} + n_2)/(n_2 - n_1)$$

$$> \tfrac{1}{2}\varepsilon\,\frac{n_1}{n_2 - n_1}\sqrt{n_1\,\mathrm{ll}\,n_1} \quad (n_1 \text{ large}).$$

Now

$$n_1/(n_2 - n_1) < n_1^{1/12}.$$

Thus

$$\tfrac{1}{2}\varepsilon\, n_1\sqrt{n_1\,\mathrm{ll}\,n_1}/(n_2 - n_1) < \tfrac{1}{2}\varepsilon n_1^{1/12}\sqrt{n_1\,\mathrm{ll}\,n_1} < \left(\sum_1^{s_{n_1}} 2pq\right)^{1/11}\sqrt{\sum_1^{s_{n_1}} 2pq} \quad (n_1 \text{ large}).$$

(1.41) is therefore applicable, and gives

$$\mathrm{B}_1(s_{n_1}, s_{n_2}) \leq \mathrm{P}(|\mu(s_{n_1})| > \tfrac{1}{2}\varepsilon\, n_1\sqrt{n_1\,\mathrm{ll}\,n_1})/(n_2 - n_1)$$

$$< \tfrac{1}{2}\exp\{-\tfrac{1}{4}\varepsilon^2 n_1^2/(n_2 - n_1)^2\,\mathrm{ll}\,n_1\,(\Sigma 2pq/n_1)^2\} < \tfrac{1}{2}\exp(-\tfrac{1}{4}\varepsilon^2 n_1/(n_2 - n_1)\,\mathrm{ll}\,n_1)$$

$$(n_1 \text{ large}).$$

The inequality

$$|\mu(s_{n_1}, s_{n_2})| > \tfrac{1}{2}\varepsilon\sqrt{n_2\,\mathrm{ll}\,n_2}$$

refers only to the set of trials from the $s_{n_1} + 1$th to the s_{n_2}th. $\Sigma 2pq$ over this set differs by less than 1 from $n_2 - n_1$, and so is large when n_1 is large. Then by (1.41)

$$\mathrm{B}_2(s_{n_1}, s_{n_2}) \leq \mathrm{P}\left(|\mu(s_{n_1}, s_{n_2})| > \tfrac{1}{2}\varepsilon\sqrt{n_2 \,\mathrm{ll}\, n_2}\right)$$
$$\leq \mathrm{P}\left(|\mu(s_{n_1}, s_{n_2})| > \tfrac{1}{2}\varepsilon \frac{\sqrt{n_2 \,\mathrm{ll}\, n_2}}{\sqrt{n_2 - n_1 + 1}} \sqrt{\sum_{s_{n_1}+1}^{s_{n_2}} 2pq}\right)$$
$$< \tfrac{1}{2}\exp\{-\tfrac{1}{4}\varepsilon^2 n_2 \,\mathrm{ll}\, n_2/(n_2 - n_1 + 1)\} < \tfrac{1}{2}\exp\{-\tfrac{1}{4}\varepsilon^2 n_1 \,\mathrm{ll}\, n_1/(n_2 - n_1)\} \quad (n_1 \text{ large}).$$

The lemma is now established.

2.3. Let us divide the events T_i, U_i whose probabilities are

$$p_1, p_2, \ldots p_n;\ q_1, \ldots q_n$$

into two classes. Consider the ith trial. Let

$$p'_i = \min(p_i, q_i).$$

Thus $p'_i \leq \frac{1}{2}$. When $p'_i < \frac{1}{2}$, one of T_i, U_i has its probability $> \frac{1}{2}$; that one is said to be the "normal event" for the ith trial, while the other is called the "abnormal event." When $p'_i = \frac{1}{2}$ we regard T_i as the normal event.

Let $\overline{m}(n)$ be the number of the p_i which are $\geq \frac{1}{2}$, *i.e.*, the number of cases in which T is the normal event. Then

$$\left|\overline{m}(n) - \sum_1^n p_i\right| = \left|\sum_{q \leq \frac{1}{2}} q_i - \sum_{p < \frac{1}{2}} p_i\right| \leq \sum_1^n 2p_i q_i. \qquad (2.31)$$

Let $m'(n)$ be the number of abnormal events which occur in the n trials, and let

$$\mu'(n) = m'(n) - \Sigma p'_i;$$

since

$$p_i q_i = p'_i q'_i$$

we have by (1.41)

$$\mathrm{P}(m'(n) > \Sigma p'_i + (\Sigma 2p_i q_i)^{7/12}) < e^{-(\Sigma 2pq)^{1/6}}$$

for sufficiently large $\sum_1^n 2pq$.

Since $\sum_1^n p_i' \leq \sum_1^n 2p_i q_i$ we infer that

$$\mathrm{P}(m'(n) > 2\Sigma 2pq) < e^{-(\Sigma 2pq)^{1/6}}, \qquad (2.32)$$

and since for a given series of events $m'(n') \leq m'(n)$ for $0 \leq n' \leq n$ we see that we can make the assertion

$$\text{“} m'(n') \leq 2\sum_1^n 2pq \quad \text{for all} \quad n' \leq n \text{”} \qquad (2.33)$$

with probability $> 1 - e^{-(\Sigma 2pq)^{1/6}}$, provided only that n is sufficiently large.

Now

$$|m(n') - \overline{m}(n')| \leq m'(n'). \qquad (2.34)$$

For

$$m(n') - \overline{m}(n') = \text{(number of T's which occur)} - \text{(total number of cases in which T is normal)}$$
$$\leq \text{(number of T's which occur)} - \text{(number of normal T's which occur)}$$
$$= \text{(number of abnormal T's which occur)}$$
$$\leq \text{(number of abnormal events which occur)}$$
$$= m'(n),$$

while

$$\overline{m}(n') - m(n') = \text{(Total number of cases in which T is normal)} - \text{(number of T's which occur)}$$
$$= \text{(total number of cases in which U is abnormal)} - \text{(number of U's which do not occur)}$$
$$\leq \text{(number of abnormal U's which occur)}$$
$$\leq m'(n').$$

From (2.31) we have

$$\left| \overline{m}(n') - \sum_1^{n'} p_i \right| \leq \sum_1^{n'} 2pq.$$

It follows that

$$\left| m(n') - \sum_1^{n'} p_i \right| \leq \sum_1^{n'} 2pq + m'(n').$$

By (2.33) we can therefore, provided only that $\sum_1^n 2pq$ is sufficiently large, make the assertion

$$\text{“}\left| m(n') - \sum_1^{n'} p_i \right| \leq 3 \sum_1^n 2pq \quad \text{for all} \quad n' \leq n\text{”} \quad . \quad . \quad . \quad . \quad . \quad (2.36)$$

with probability $> 1 - \exp\{-(\Sigma 2pq)^{1/6}\}$.

Consider now the sequence of trials from the $s_{i^2}+1$th to the $s_{(i+1)^2}$th, i being a positive integer. $\Sigma 2pq$ over this sequence lies between $2i$ and $2i+2$. It is therefore large when i is large. We infer that if i is large we can assert with probability greater than $1 - \exp\{-(2i)^{1/6}\}$ that for every n' in $(s_{i^2}+1, s_{(i+1)^2})$

$$|\mu(s_{i^2}, n')| \leq 6(i+1),$$

and, since

$$6(i+1)/\chi(s_{i^2}) < 6(i+1)/\sqrt{i^2 \, \mathrm{ll}\, i^2} < \varepsilon \quad \text{for} \quad i \geq i_0(\varepsilon),\ n' \geq s_{i^2}+1,$$

we have:

“If $\varepsilon > 0$, then when $i \geq i_0(\varepsilon)$ we can assert with probability greater than $1 - \exp\{-(2i)^{1/6}\} = 1 - x_{n'}$ that for every n' in $(s_{i^2}+1, s_{(i+1)^2})$

$$|\mu(s_{i^2}, n')|/\chi(s_{i^2}) < \varepsilon.\text{”} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.37)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

2.4. Now let $0 < \tau < 1$, and let

$$n_{m,\kappa} = [(1+\tau)^m (1 + k\tau/m)] \qquad (k = 0, 1, \dots m-1, m).$$

Thus $n_{m,0} = [(1+\tau)^m]$, the greatest integer $\leq (1+\tau)^m$, and therefore, writing

$$A(s_{n_{m,0}}) = P(|\mu(s_{n_{m,0}})|/\chi(s_{n_{m,0}}) > 1 + \varepsilon) \qquad (0 < \varepsilon < 1),$$

we have by (1.41)

$$\begin{aligned} A(s_{n_{m,0}}) &= P(|\mu(s_{n_{m,0}})| > (1+\varepsilon)\chi(s_{n_{m,0}})) \\ &< \exp\{-(1+\varepsilon)^2 \,\mathrm{ll}\, n_{n_{m,0}} - \log 2\} \\ &< \exp\{-(1+\varepsilon)^2 \,\mathrm{ll}\,(1+\tau)^m\} \qquad \text{for } m \geq m_0(\tau) \\ &= [m\,\mathrm{l}\,(1+\tau)]^{-(1+\varepsilon)^2} = u_m, \text{ say.} \qquad (2.41) \end{aligned}$$

Next let $n_1 = n_{m,0}$, $n_2 = n_{m,\kappa}$ $(\kappa \geq 1)$; then for $m \geq m_0(\tau)$

$$n_1^{11/12} < n_2 - n_1 < n_1.$$

For

(1) $n_2 - n_1 < n_{m,0}\dfrac{\kappa\tau}{m} + 1 \leq n_{m,0}\tau + 1 < n_{m,0}$ (since $\tau < 1$) for $m \geq m_0(\tau)$;

(2) $n_2 - n_1 > n_{m,0}\dfrac{\kappa\tau}{m} - 1 > \tfrac{1}{2}n_{m,0}\dfrac{\kappa\tau}{m} = n_{m,0}^{11/12}\cdot\dfrac{n_{m,0}^{1/12}\kappa\tau}{2m} > n_{m,0}^{11/12}$ for $m \geq m_0(\tau)$.

By (2.22) it therefore follows that for $\kappa \geq 1$, $m \geq m_0(\tau)$

$$B(s_{n_{m,0}}, s_{n_{m,\kappa}}) < \exp\left(-\frac{\varepsilon^2}{4}\frac{n_{m,0}\,\mathrm{ll}\,n_{m,0}}{n_{m,0}\dfrac{\kappa\tau}{m}+1}\right) < \exp\left(-\frac{\varepsilon^2}{5}\frac{m}{\kappa\tau}\,\mathrm{ll}\,n_{m,0}\right) < \exp\left(-\frac{\varepsilon^2}{6\tau}\,\mathrm{ll}\,(1+\tau)^m\right)$$

$$= \{m \log(1+\tau)\}^{-\frac{\varepsilon^2}{6\tau}},$$

whence

$$\sum_{\kappa=1}^{m-1} B(s_{n_{m,0}}, s_{n_{m,\kappa}}) < m^{1-\frac{\varepsilon^2}{6\tau}}\{\log(1+\tau)\}^{-\frac{\varepsilon^2}{6\tau}}.$$

We now take $\tau = \dfrac{\varepsilon^2}{18}$. Thus $\dfrac{\varepsilon^2}{6\tau} = 3$ and

$$\begin{aligned} \sum_{\kappa=1}^{n-1} B(s_{n_{m,0}}, s_{n_{m,\kappa}}) &< \frac{1}{m^2}\left\{\log\left(1+\frac{\varepsilon^2}{18}\right)\right\}^{-3} \quad \text{for all } m \geq m_0(\varepsilon) \\ &= v_m, \text{ say.} \qquad (2.42) \end{aligned}$$

The numbers $n_{m,\kappa}$ divide up the range of positive integers into segments, and since

$$\dots n_{m-1,m-1} = n_{m,0} < n_{m,1} < n_{m,2} < \dots < n_{m,m-1} < n_{m,m} = n_{m+1,0} < \dots$$

(provided m is sufficiently large) we see that, given i, the inequalities

$$i \geq i_0, \quad 0 \leq \kappa \leq m-1,$$
$$n_{m,\kappa} \leq i^2 < n_{m,\kappa+1} \quad \ldots \quad (2.43)$$

determine m, κ uniquely.

Let us agree to interpret $n_{m,-1}$ as $n_{m-1,m-2}$. Then we may speak of $\mathrm{B}(s_{n_{m,\kappa-1}}, s_{i^2})$. As before, if $n_1 = n_{m,\kappa-1}$, $n_2 = i^2$ we have easily

$$n_1^{11/12} < n_2 - n_1 < n_1,$$

when $i \geq i_0$. Lemma (2.22) is therefore applicable, and we have

$$\mathrm{B}(s_{n_{m,\kappa-1}}\, s_{i^2}) < \exp\{-\tfrac{1}{4}\varepsilon^2\, n_{m,\kappa-1} \,\mathrm{ll}\, n_{m,\kappa-1}/(n_{m,\kappa} - n_{m,\kappa-1})\},$$

and consequently

$$\mathrm{C}(s_{n_{m,\kappa}}, s_{n_{m,\kappa+1}}) = \sum_{(n_{m,\kappa} \leq i^2 < n_{m,\kappa+1})} \mathrm{B}(s_{n_{m-1,\kappa}}, s_{i^2})$$
$$< (n_{m,\kappa+1} - n_{m,\kappa}) \exp\left(-\tfrac{1}{4}\varepsilon^2 \frac{n_{m,\kappa-1} \,\mathrm{ll}\, n_{m,\kappa-1}}{n_{m,\kappa} - n_{m,\kappa-1}}\right)$$
$$< n_{m,0} \exp\left(-\tfrac{1}{4}\varepsilon^2 \frac{n_{m,\kappa-1} \,\mathrm{ll}\, n_{m,\kappa-1}}{n_{m,\kappa-1}\dfrac{\tau}{m-1}+1}\right)$$
$$< \exp\left(-\tfrac{1}{4}\varepsilon^2 \frac{n_{m,\kappa-1} \,\mathrm{ll}\, n_{m,\kappa-1}}{\dfrac{2\tau}{m} n_{m,\kappa-1}} + m \log(1+\tau)\right)$$
$$= \exp\left(-m\left[\frac{\varepsilon^2}{8\tau} \,\mathrm{ll}\, n_{m,\kappa-1} - \log(1+\tau)\right]\right) < e^{-m}. \quad \ldots \quad (2.44)$$

Whence

$$\sum_{\kappa=0}^{m-1} \mathrm{C}(s_{n_{m,\kappa}}, s_{n_{m,\kappa+1}}) < me^{-m} = w_m, \quad \ldots \quad (2.45)$$

for $m \geq m_0(\varepsilon)$, or (since m is large when i is large) for $i \geq i_0(\varepsilon)$.

If we take the m of (2.41), (2.42) to be defined by (2.43) in terms of i, we see that these inequalities also are valid for all sufficiently large i.

2.5. Now the series Σu_m, Σv_m, Σw_m, Σx_m are convergent series of positive terms. Given $\eta > 0$ we can therefore choose $m_1(\eta, \varepsilon)$ so big that

(1) (2.37) is valid for $i^2 \geq m_1$ and $e^{-(2i)^{1/6}} < \varepsilon$,

(2) (2.41), (2.42), (2.45) are valid for $m \geq m_1$,

(3) $\sum_{m_1}^{\infty} (u_m + v_m + w_m + x_m) \leq \tfrac{1}{2}\eta$.

Let n' be any positive integer, and let i be defined by

$$s_{i^2} < n' \leq s_{(i+1)^2}.$$

Thus (1), (2), (3) hold if $n' \geq n_1(\eta, \varepsilon)$.
Then :

The probability that

(a) $$|\mu(s_{n_{m,0}})/\chi(s_{n_{m,0}})| \geq 1 + \varepsilon$$

for at least one $m \geq m_1$ is not greater than

$$\sum_{m_1}^{\infty} A(s_{n_{m,0}}) < \sum_{m_1}^{\infty} u_m;$$

the probability that for some pair of integers m, k such that $m \geq m_1$, $1 \leq k \leq m$

(b) $$|\mu(s_{n_{m,0}})/\chi(s_{n_{m,0}}) - \mu(s_{n_{m,\kappa}})/\chi(s_{n_{m,\kappa}})| \geq \varepsilon$$

is not greater than

$$\sum_{m=m_1}^{\infty} \sum_{k=1}^{m} B(s_{n_{m,0}}, s_{n_{m,\kappa}}) < \sum_{m_1}^{\infty} v_m;$$

the probability that for at least one $i^2 \geq n_{m,0}$, where m, k are given by

$$n_{m,\kappa} \leq i^2 < n_{m,\kappa+1},$$

the inequality

(c) $$|\mu(s_{i^2})/\chi(s_{i^2}) - \mu(s_{n_{m,\kappa}})/\chi(s_{n_{m,\kappa}})| \geq \varepsilon$$

is satisfied is not greater than

$$\sum_{m=m_1}^{\infty} \sum_{k=u}^{m-1} C(s_{n_{m,\kappa}}, s_{n_{m,\kappa+1}}) < \sum_{m_1}^{\infty} w_m;$$

and the probability that for at least one pair n', i, where $n' \geq n_1$, $s_{i^2} < n' \leq s_{(i+1)^2}$,

(d) $|\mu(n')/\chi(s_{i^2}) - \mu(s_{i^2})/\chi(s_{i^2})| \geq \varepsilon$ is not greater than $\sum_{m=m_1}^{\infty} x_m$.

We can therefore assert with probability greater than $1 - \sum_{m=m_1}^{\infty}(u_m + v_m + w_m + x_m)$ that for no $n' \geq n_1$ is any of (a), (b), (c), (d) satisfied.

Now

$$\left|\frac{\mu(n')}{\chi(n')}\right| \leq \left|\frac{\mu(n')}{\chi(s_{i^2})}\right|$$

and

$$\left|\frac{\mu(n')}{\chi(s_{i^2})}\right| \leq \left|\frac{\mu(s_{n_{m,0}})}{\chi(s_{n_{m,0}})}\right| + \left|\frac{\mu(s_{n_{m,\kappa}})}{\chi(s_{n_{m,\kappa}})} - \frac{\mu(s_{n_{m,0}})}{\chi(s_{n_{m,0}})}\right| + \left|\frac{\mu(s_{i^2})}{\chi(s_{i^2})} - \frac{\mu(s_{n_{m,\kappa}})}{\chi(s_{n_{m,\kappa}})}\right| + \left|\frac{\mu(n')}{\chi(s_{i^2})} - \frac{\mu(s_{i^2})}{\chi(s_{i^2})}\right|.$$

When none of (a), (b), (c), (d) is satisfied, therefore,

$$\mu(n')/\chi(n') < (1+\varepsilon) + \varepsilon + \varepsilon + \varepsilon = 1 + 4\varepsilon = 1 + \delta.$$

Thus

"Given $\delta > 0$, $\eta > 0$ we can find $n_1 = n_1(\eta, \delta)$ so that the truth for all $n \geq n_1$ of

$$|\mu(n)/\chi(n)| < 1 + \delta$$

may be asserted with probability greater than $1 - \frac{1}{2}\eta$." . . . . (2.51)

This completes the first stage of our discussion; that relating to (1) of (2.12).

2.61 *Lemma.*—For

$$0 < \varepsilon < 1, \quad n \geq n_0(\varepsilon), \quad \mathrm{P}[\,|\mu(n)/\chi(n)| > (1-\varepsilon)\,] > \frac{1}{\left(\log \sum_1^n 2pq\right)^{1-\varepsilon}}.$$

Proof.

In (A) we put $\varepsilon = \frac{1}{12}$, and suppose t_1, t_2 defined* by the equations

$$\begin{aligned} t'_1 &= (-1+\varepsilon)\sqrt{11\,\Sigma 2pq}, & t'_2 &= (1-\varepsilon)\sqrt{11\,\Sigma 2pq}, \\ m'_1 &= \Sigma p + t'_1\sqrt{\Sigma 2pq}, & m'_2 &= \Sigma p + t'_2\sqrt{\Sigma 2pq}, \\ m_1 &= [m'_1], & m_2 &= -[-m'_2], \\ m_1 &= \Sigma p + t_1\sqrt{\Sigma 2pq}, & m_2 &= \Sigma p + t_2\sqrt{\Sigma 2pq}. \end{aligned}$$

Let $\lambda_1 = m'_1 - m_1$, $\lambda_2 = m_2 - m'_2$. Thus $0 \leq \lambda_1, \lambda_2 < 1$. Then

$$\begin{aligned} t_1 - 1/2\sqrt{\Sigma 2pq} &= t'_1 - (\tfrac{1}{2} - \lambda_1)/\sqrt{\Sigma 2pq} = t'_1 + \mu_1/2\sqrt{\Sigma 2pq}, \\ t_2 + 1/2\sqrt{\Sigma 2pq} &= t'_2 + (\tfrac{1}{2} - \lambda'_2)/\sqrt{\Sigma 2pq} = t'_2 + \mu_2/2\sqrt{\Sigma 2pq}, \end{aligned}$$

where $|\mu_1| \leq 1$, $|\mu_2| \leq 1$.

By (A) we then have, since $e^{-t^2}(3t - 2t^3)$ is an odd function,

$$\begin{aligned} \mathrm{P}(m'_1, m'_2) = \mathrm{P}(m_1, m_2) &= \frac{1}{\sqrt{\pi}} \int_{(-1+\epsilon)\sqrt{11\,\Sigma 2pq} + \frac{\mu_1}{2\sqrt{\Sigma 2pq}}}^{(1-\epsilon)\sqrt{11\,\Sigma 2pq} + \frac{\mu_1}{2\sqrt{\Sigma 2pq}}} e^{-t^2}\,dt \\ &\quad + \theta_{21}\frac{1}{6\sqrt{\pi}\sqrt{\Sigma 2pq}} \int_{(1-\epsilon)\sqrt{11\,\Sigma 2pq} - \frac{\mu_1}{2\sqrt{\Sigma 2pq}}}^{(1-\epsilon)\sqrt{11\,\Sigma 2pq} + \frac{\mu_2}{2\sqrt{\Sigma 2pq}}} e^{-t^2}(3t - 2t^3)\,dt \\ &\quad + \theta_{20}\left[(\Sigma 2pq)^{-1/2} + \tfrac{9}{8}\Sigma 2pq\, e^{-1/2(\Sigma pq)^{1/6}}\right], \quad \text{for } n \geq n_2(\varepsilon). \end{aligned}$$

We can write the last term as $2\theta_{22}/\sqrt{\Sigma 2pq}$ if $n_2(\varepsilon)$ is chosen large enough. And when n is large

$$|3t - 2t^3| < 2t^3\sqrt{\pi},$$

* The ε of (A) has already been chosen equal to $\frac{1}{12}$, so that there is no ambiguity.

so that the second term is absolutely less than

$$\frac{1}{3\sqrt{\Sigma 2pq}}\int_{(1-\epsilon)\sqrt{\text{ll }\Sigma 2pq}-\frac{1}{2\sqrt{\Sigma 2pq}}}^{(1-\epsilon)\sqrt{\text{ll }\Sigma 2pq}+\frac{1}{2\sqrt{\Sigma 2pq}}} e^{-t^2}t^3\,dt$$

$$< \frac{1}{3\sqrt{\Sigma 2pq}}\int_{(1-\epsilon)\sqrt{\text{ll }\Sigma 2pq}}^{(1-\epsilon)\sqrt{\text{ll }\Sigma 2pq}+\frac{1}{\sqrt{\Sigma 2pq}}} e^{-u^2+1}\,(u - 1/2\sqrt{\Sigma 2pq})^3\,du, \quad (n \text{ large})$$

$$< \frac{1}{\Sigma 2pq}(\log \Sigma 2pq)^{-(1-\epsilon)^2}(\text{ll }\Sigma 2pq)^3$$

since the integrand is a decreasing function when n is large

$$< \frac{1}{\Sigma 2pq}.$$

Thus

$$1 - \mathrm{P}(m'_1, m'_2) \geq \frac{2}{\sqrt{\pi}}\int_{(1-\epsilon)\sqrt{\text{ll }\Sigma 2pq}+\frac{1}{2\sqrt{\Sigma 2pq}}}^{\infty} e^{-t^2}\,dt - \frac{2}{\sqrt{\Sigma 2pq}} - \frac{1}{\Sigma 2pq}$$

$$> \frac{2}{\sqrt{\pi}}\int_{(1-\epsilon)\sqrt{\text{ll }\Sigma 2pq}}^{\infty} e^{-t^2}\,dt - \frac{4}{\sqrt{\Sigma 2pq}}.$$

Now

$$\int_{t_0}^{\infty} e^{-t^2}\,dt > \int_{t_0}^{t_0+\frac{1}{4t_0}} e^{-t^2}\,dt > \frac{1}{4t_0}e^{-t_0^2-\frac{1}{2}-\frac{1}{16t_0^2}} > \frac{e^{-t_0^2}}{8t_0}\cdot (t_0 \geq \tfrac{1}{2}).$$

It follows that for $n \geq n_0(\varepsilon)$,

$$1 - \mathrm{P}(m'_1, m'_2) > \frac{1}{4\sqrt{\pi}}\,\frac{1}{(1-\varepsilon)\sqrt{\text{ll }\Sigma 2pq}\,(\log \Sigma 2pq)^{(1-\epsilon)^2}} - \frac{4}{\sqrt{\Sigma 2pq}} > \frac{1}{\left(\log \sum_1^n 2pq\right)^{1-\epsilon}}.$$

2.62. Let $\mathrm{A} = \mathrm{A}(a, \delta, \eta)$ be a positive integer, which will be chosen more precisely later, and let P_κ denote the probability that the κ inequalities

$$\begin{cases} |\mu(s_{\mathrm{A}^\kappa})/\chi(s_{\mathrm{A}^\kappa})| > 1 - 2\varepsilon, & \qquad (2.621) \\ |\mu(s_{\mathrm{A}^i})/\chi(s_{\mathrm{A}^i})| \leq 1 - 2\varepsilon \quad \text{for} \quad i = 1, 2, \ldots \kappa - 1 & \qquad (2.622) \end{cases}$$

are simultaneously true. Then $\sum_1^t \mathrm{P}_\kappa$ is the probability that (2.621) is satisfied for at least one $k \leq t$.

Consider the set of trials from the $(s_{\mathrm{A}^{\kappa-1}} + 1)$th to the s_{A^κ}th; let the event T appear

$$\mathrm{M} = m(s_{\mathrm{A}^\kappa}) - m(s_{\mathrm{A}^{\kappa-1}}) \text{ times.}$$

Then the probability π_κ of

$$\left|\mathrm{M} - \sum_{s_{\mathrm{A}^{\kappa-1}}+1}^{s_{\mathrm{A}^\kappa}} p\right| \Big/ \sqrt{\sum_{s_{\mathrm{A}^{\kappa-1}}+1}^{s_{\mathrm{A}^\kappa}} 2pq \text{ ll} \sum_{s_{\mathrm{A}^{\kappa-1}}+1}^{s_{\mathrm{A}^\kappa}} 2pq} > 1 - \varepsilon \quad . \quad . \quad . \quad (2.623)$$

satisfies, by lemma 2.61, the inequality

$$\pi_\kappa > 1/[\log (A^\kappa - A^{\kappa-1} - 1)]^{1-\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.624)$$

for all $k \geq 1$ provided only that A is sufficiently large.

Suppose now that (2.623) holds for a particular k and that (2.622) holds for every $i < k$. We shall show that (2.621) then holds for this k. We have

$$M - \sum_{s_{A^{\kappa-1}+1}}^{s_{A^\kappa}} p = \mu(s_{A^\kappa}) - \mu(s_{A^{\kappa-1}});$$

therefore if (2.623) holds we have

$$|\mu(s_{A^\kappa})| > (1-\varepsilon)\sqrt{\sum_{s_{A^{\kappa-1}+1}}^{s_{A^\kappa}} 2pq \, ll(A^\kappa - A^{\kappa-1} - 1)} - |\mu(s_{A^{\kappa-1}})|.$$

Now

$$\sum_{s_{A^{\kappa-1}+1}}^{s_{A^\kappa}} 2pq \Big/ \sum_1^{s_{A^\kappa}} 2pq = 1 - \sum_1^{s_{A^{\kappa-1}}} 2pq \Big/ \sum_1^{s_{A^\kappa}} 2pq \geq 1 - (A^{\kappa-1}+1)/A^\kappa > 1 - 2/A.$$

Therefore

$$|\mu(s_{A^\kappa})| > (1-\varepsilon)\sqrt{1-\frac{2}{A}}\,\chi(s_{A^\kappa})\sqrt{\frac{ll(A^\kappa - A^{\kappa-1} - 1)}{ll(A^\kappa+1)}} - |\mu(s_{A^{\kappa-1}})|.$$

But by (2.622)

$$|\mu(s_{A^{\kappa-1}})| < (1-2\varepsilon)\chi(s_{A^{\kappa-1}}) < (1-2\varepsilon)\left(\sum_1^{s_{A^{\kappa-1}}} 2pq \Big/ \sum_1^{s_{A^\kappa}} 2pq\right)^{1/2} \chi(s_{A^\kappa})$$

$$< (1-2\varepsilon)\left(\frac{A^{\kappa-1}+1}{A^\kappa}\right)^{1/2}\chi(s_{A^\kappa})$$

$$< (1-\varepsilon)\chi(s_{A^\kappa})/\sqrt{A} \qquad (k = 1, 2, \ldots, \text{A large}).$$

Thus finally

$$|\mu(s_{A^\kappa})| > \chi(s_{A^\kappa})\left\{(1-\varepsilon)\sqrt{1-\frac{2}{A}}\sqrt{\frac{ll(A^\kappa - A^{\kappa-1} - 1)}{ll(A^\kappa+1)}} - (1-\varepsilon)\frac{1}{\sqrt{A}}\right\}.$$

(2.621) will follow *a fortiori* if the expression in curly brackets is $> 1 - 2\varepsilon$; this will be so for all values of $k = 1, 2, \ldots$ if only A is sufficiently large.

(2.621) therefore follows from (2.622) and (2.623); P_κ is consequently not less than the probability that (2.622) and (2.623) are simultaneously true. The probability that (2.622) are all satisfied is

$$1 - \sum_1^{\kappa-1} P_i.$$

Further the trials to which (2.622) and (2.623) refer are independent of one another. Thus

$$P_\kappa \geq \pi_\kappa\left(1 - \sum_1^{\kappa-1} P_i\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.625)$$

Now by (2.624)

$$\pi_\kappa > 1/[\log(A^\kappa - A^{\kappa-1} - 1)]^{1-\epsilon} \qquad (k = 1, 2, \ldots \infty)$$
$$> \gamma(A, \varepsilon)/\kappa^{1-\epsilon} \qquad \text{where } \gamma > 0 \text{ is independent of } \kappa.$$

Thus $\sum_1^\infty \pi_\kappa$ diverges.

But from the definition of P_κ it is plain that $\sum_1^\infty P_\kappa$ converges. From (2.625) we therefore have

$$\lim_{\kappa\to\infty}\left(1 - \sum_1^{\kappa-1} P_i\right) = 0,$$

and hence K can be chosen so that $\sum_1^K P_i > 1 - \eta/2$. We can then assert with probability $> 1 - \eta/2$ that (2.621) holds for at least one $k \leq K$. Let us choose A greater than the n_1 of (2.51). Then combining our present result with (2.51) we have :

" Given $\delta > 0$, $\eta > 0$ we can find a positive integer $A = A(\eta, \delta)$, as large as we please, such that it is possible to make with probability $> 1 - \eta$ the double assertion

1. $|\mu(n)| < (1 + \delta)\chi(n)$ for all $n \geq A$
2. $|\mu(n)| > (1 - \delta)\chi(n)$ for at least one $n \geq A$."

(2.12) is therefore established.

3. We now prove a theorem which shows that the result just obtained can be improved upon. The theorem asserts that given an infinite sequence of probabilities and an arbitrary function $f(n)$, the probability that

$$|\mu(n)| < f(n) \text{ for all large } n$$

(*i.e.*, for all n from some point, no matter what, onwards) is either 0 or 1. Naturally this involves an extension of the definition of a probability ; the details of this extension are given in (3.1).

We can restate KHINTCHINE's result as follows :

" The probability that, given $\delta > 0$,

$$\begin{cases} |\mu(n)| < (1 + \delta)\chi(n) \text{ for all large } n \\ |\mu(n)| > (1 - \delta)\chi(n) \text{ for an infinity of } n \end{cases}$$

is 1."

The theorem mentioned above shows, on putting $f(n) = \chi(n)$ that :

" One of $P(|\mu(n)| < \chi(n)$ for all large $n)$, $P(|\mu(n)| \geq \chi(n)$ for an infinity of $n)$ is 1."

Combining these we see that :

" We can, given $\delta > 0$, assert with probability 1 that either :

$|\mu(n)| < (1 + \delta)\chi(n)$ for all large n, and $|\mu(n)| \geq \chi(n)$ for an infinity of n,

or

$|\mu(n)| < \chi(n)$ for all large n, and $|\mu(n)| > (1 - \delta)\chi(n)$ for an infinity of n."

We shall return to this point in 3.5.

3.1. Let $\{p_i\}$ be a given infinite sequence of probabilities such that $\sum_1^n 2p_i q_i = \varepsilon(n) \to \infty$ as $n \to \infty$. Let $f(n)$ be any positive increasing function of n.

Consider the probability that

$$|\mu(n)| < f(n)$$

for all values of n from a certain point (no matter what) onwards. We shall show that for any given $f(n)$ the probability is either 0 or 1.

We can define this probability as a double limit as follows. The probability that

$$|\mu(n)| < f(n) \text{ for all } n \text{ in } (n_0, \mathrm{N})$$

is, by definition, the sum of a certain finite set of terms of the form $p_1\, q_2\, q_3\, p_4\, q_5 \dots p_\mathrm{N}$ selected from the set of all such terms, which contains 2^N members. Denoting this probability by $\mathrm{P}(n_0, \mathrm{N})$ and observing that it decreases as $\mathrm{N} \to \infty$, remaining always ≥ 0, we may define the probability that

$$|\mu(n)| < f(n) \text{ for all } n \geq n_0$$

as

$$\mathrm{P}(n_0) = \lim_{\mathrm{N} \to \infty} \mathrm{P}(n_0, \mathrm{N}).$$

Now increasing n_0 increases (in the wide sense) $\mathrm{P}(n_0, \mathrm{N})$.
Thus for every $\mathrm{N} \geq n_0 + 1$,

$$\mathrm{P}(n_0 + 1, \mathrm{N}) \geq \mathrm{P}(n_0, \mathrm{N}),$$

whence

$$\mathrm{P}(n_0 + 1) \geq \mathrm{P}(n_0).$$

But $\mathrm{P}(n_0) \leq 1$ always. Therefore $\mathrm{P}(n_0) \to \lim$ as $n_0 \to \infty$, and we may define the probability that

$$|\mu(n)| < f(n) \text{ for all } n \text{ from some } n \text{ or other onwards}$$

as $\mathrm{P} = \lim_{n_0 \to \infty} \mathrm{P}(n_0)$. So that

$$\mathrm{P} = \lim_{n \to \infty} \lim_{\mathrm{N} \to \infty} \mathrm{P}(n, \mathrm{N}). \qquad (3.11)$$

We infer that, given $0 < \varepsilon_0 < 1$, we can find n_0, $\mathrm{N}_0 > n_0$ such that

$$|\mathrm{P} - \mathrm{P}(n, \mathrm{N})| < \varepsilon_0$$

for all $n \geq n_0(\varepsilon_0)$ and all $\mathrm{N} \geq \mathrm{N}_0(n, \varepsilon_0)$.

Let $n' > n_0$ and $\mathrm{N}' > \mathrm{N}_0(n', \varepsilon_0)$; let $n'' > \mathrm{N}'$ and $\mathrm{N}'' > \mathrm{N}_0(n'', \varepsilon_0)$.

Then

$$\mathrm{P}(n', \mathrm{N}') = \mathrm{P} + \theta_1\, \varepsilon_0$$
$$\mathrm{P}(n', \mathrm{N}'') = \mathrm{P} + \theta_2\, \varepsilon_0$$
$$\mathrm{P}(n'', \mathrm{N}'') = \mathrm{P} + \theta_3\, \varepsilon_0.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Now

$$\begin{aligned} P(n', N'') &= P(|\mu(n)| < f(n) \text{ throughout } (n', N'')) \\ &\leq P(|\mu(n)| < f(n) \text{ throughout } (n', N') \text{ and } (n'', N'')) \\ &= P(n', N') \,.\, P(\mu(n) < f(n) \text{ in } (n'', N''), \text{ given this in } (n', N')). \quad (3.12) \end{aligned}$$

We proceed to investigate this last probability, and to show that (n', N') being fixed, it is arbitrarily near to $P(n'', N'')$ for all sufficiently large n''.

3.21 *Lemma.*

" For $\sum_1^n 2pq > K$ (an absolute constant), and for all values of m,

$$P(m, n) < \frac{1}{\left(\sum_1^n 2pq\right)^{1/3}}."$$

Proof.

$$P(m, n) = \frac{1}{2\pi i} \int_{(|x|=1)} \frac{\prod_1^n (px+q)}{x^{m+1}} dx = \frac{1}{2\pi i}\int_{-\phi_0}^{\phi_0} + \frac{1}{2\pi i}\int_{\phi_0}^{2\pi-\phi_0}$$

We choose

$$\phi_0 = \sqrt{2}\sqrt{\log \sum_1^n 2pq} \Big/ \sqrt{\sum_1^n 2pq}.$$

Then

$$\left|\frac{1}{2\pi i}\int_{-\phi_0}^{\phi_0}\right| < \frac{2\phi_0}{2\pi} < 2\frac{1}{\left(\sum_1^n 2pq\right)^{1/3}} \quad \text{for all large } \sum_1^n 2pq;$$

$$\left|\frac{1}{2\pi i}\int_{\phi_0}^{2\pi-\phi_0}\right| \leq \frac{1}{2\pi}\int_{\phi_0}^{2\pi-\phi_0}\left|\prod_1^n (px+q)\right| d\phi$$

$$= \left|\prod_1^n (pe^{i\phi_0}+q)\right| = \exp\left(0\cdot051\,\theta_6\phi_0^4\,\Sigma pq - \tfrac{1}{2}\phi_0^2\,\Sigma pq\right) \text{ by (C)}$$

$$= \exp\left\{0\cdot051\,\theta_6 2(\log \Sigma 2pq)^2/\Sigma 2pq\right\} \Big/ \sqrt{\sum_1^n 2pq}$$

$$< 1 \Big/ 2\left(\sum_1^n 2pq\right)^{1/3} \text{ for all large } \sum_1^n 2pq.$$

The lemma is therefore proved.

3.22 *Lemma.*

" For $n_0 \geq 1$, $n \geq N_1(n_0)$ and for all m_1, m_2

$$|P(m_1 \leq m \leq m_2, n) - P^*(m_1 \leq m \leq m_2, n - n_0)| > \left(\sum_1^n 2pq\right)^{-1/3}."$$

Here P^* refers to the set of trials from the $n_0 + 1$th to the nth.

Proof.

$$\mathrm{P}^*(m_1 \leq m \leq m_2,\ n - n_0) - \mathrm{P}(m_1 \leq m \leq m_2,\ n)$$

$$= \text{coefficient of } x^{m_2} \text{ in } \frac{1 - x^{m_2 - m_1 + 1}}{1 - x}\left(1 - \prod_1^{n_0}(px+q)\right)\prod_{n_0+1}^{n}(px+q)$$

$$= \frac{1}{2\pi i}\int_{\mathrm{C}} \frac{1 - \prod_1^{n_0}(px+q)}{1-x}\,(1 - x^{m_2 - m_1 + 1})\prod_{n_0+1}^{n}(px+q)\,\frac{dx}{x^{m_2+1}},$$

where C is the circle $|x| = 1$.

Now

$$\frac{1 - \prod_1^{n_0}(px+q)}{1-x} = \frac{1 - \prod_1^{n_0}(1 - p(1-x))}{1-x} = \frac{1 - e^{\sum_1^{n_0}\log(1-pt)}}{t}$$

(writing t for $1 - x$).

When $|t| < 1$,

$$\log(1 - pt) = -pt - \frac{p^2t^2}{2} - \frac{p^3t^3}{3} - \dots .$$

For

$$|t| < \frac{1}{n_0+1}, \quad n_0 \geq 1,$$

therefore,

$$\log(1-pt) = -pt - \frac{\theta_1 t}{n_0},$$

and

$$1 - e^{\sum_1^{n_0}\log(1-pt)} = 1 - e^{-t\sum_1^{n_0}p - \theta_2 t}$$

$$= 1 - e^{\theta_3 t(n_0+1)}.$$

Again, when

$$|u| < \frac{1}{n_0+2},\ n_0 \geq 1,$$

$$|1 - e^u| = \left|u + \frac{u^2}{2\,!} + \frac{u^3}{3\,!}\dots\right| = |u|\left(1 + \frac{\theta_4}{n_0+1}\right).$$

Thus for

$$|t| < \frac{1}{(n_0+1)(n_0+2)},$$

$$\left|1 - e^{\sum_1^{n_0}\log(1-pt)}\right| < (n_0+2)_1|t|.$$

We have now proved that

$$\left|\frac{1 - \prod_1^{n_0}(px+q)}{1-x}\right| < n_0 + 2 \quad \dots\dots\dots \quad (3.221)$$

for

$$|1-x| < \frac{1}{(n_0+1)(n_0+2)}.$$

Let us write $x = e^{i\phi}$; then

$$\int_C = \int_{\phi=0}^{2\pi} = \int_{-\phi_0}^{\phi_0} + \int_{\phi_0}^{2\pi-\phi_0},$$

where we choose

$$\phi_0 = \sqrt{2}\sqrt{\log \sum_{n_0+1}^{n} 2pq} \Big/ \sqrt{\sum_{n_0+1}^{n} 2pq}.$$

Then

$$\left|\frac{1}{2\pi i}\int_{-\phi_0}^{\phi_0}\right| \leq \frac{1}{2\pi}\int_{-\phi_0}^{\phi_0}\left|\frac{1-\prod_{1}^{n_0}(px+q)}{1-x}\right| . |1-x^{m_2-m_1+1}| . \left|\prod_{n_0+1}^{n}(px+q)\right| . \frac{|dx|}{|x^{m_2+1}|}.$$

$$< \frac{1}{2\pi} . 2\phi_0 . (n_0+2) 2 . 2\pi \quad . \quad . \quad . \quad . \quad (3.222)$$

provided $\phi_0 < \dfrac{1}{(n_0+1)(n_0+2)}$. (For $|1-x| < |\phi|$.) This is so for $n \geq A_1(n_0)$.

Also

$$\left|\frac{1}{2\pi i}\int_{+\phi_0}^{2\pi-\phi_0}\right| \leq (n_0+2) . 2 . \frac{1}{2\pi}\int_{\phi_0}^{2\pi-\phi_0} \left|\prod_{n_0+1}^{n}(px+q)\right| d\phi.$$

The integrand is greatest at ϕ_0. Thus

$$\left|\frac{1}{2\pi i}\int_{\phi_0}^{2\pi-\phi_0}\right| < (n_0+2) . 2 . \left|\prod_{n_0+1}^{n}(pe^{i\phi_0}+q)\right|$$

$$= 2(n_0+2)\exp\left(0{\cdot}051\,\theta_6\phi_0^4\sum_{n_0+1}^{n} pq - \tfrac{1}{2}\phi_0^2\sum_{n_0+1}^{n} pq\right)$$

by (C); the condition $\phi_0 \leq \frac{1}{2}$ is satisfied for all $n \geq A_2(n_0)$;

$$= 2(n_0+2)\,e^{0{\cdot}051\theta_6} . \frac{4\left(\log \sum_{n_0+1}^{n} 2pq\right)^2}{2 . \sum_{n_0+1}^{n} 2pq}\exp\left(-\tfrac{1}{2}\log\sum_{n_0+1}^{n} 2pq\right)$$

$$< 3(n_0+2)\Big/\sqrt{\sum_{n_0+1}^{n} 2pq} \quad \text{for } n \geq A_3(n_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.223)$$

Since

$$4(n_0+2)\phi_0 + 3(n_0+2)\Big/\sqrt{\sum_{n_0+1}^{n} 2pq} < 1\Big/\left(\sum_{1}^{n} 2pq\right)^{1/3}$$

for all sufficiently large n, the lemma now follows from (3.222) and (3.223).

3.23 *Lemma.*

"If $n_0 \geq 1$, $0 \leq m_0 \leq n_0$, $\mu_0 = m_0 - \sum_1^{n_0} p$, and $\varepsilon > 0$ is arbitrary, then

$$|\,P(|\mu(n)| < f(n)) - P^*(-f(n) - \mu_0 < \mu(n_0, n) < f(n) - \mu_0)| < \varepsilon$$

for all $n \geq N_2(n_0, \varepsilon)$."

P^*, as before, refers to the set $(n_0 + 1, n)$ of trials.

Proof.

By lemma 3.22

$$|\,P(-f(n) < \mu(n) < f(n)) - P^*(-f(n) < \mu(n_0, n) < f(n))\,| < \tfrac{1}{2}\varepsilon$$

for all $n \geq N_1(n_0, \varepsilon)$.

But

$$|\,P^*(-f(n) < \mu(n_0, n) < f(n)) - P^*(-f(n) - \mu_0 < \mu(n_0, n) < f(n) - \mu_0)\,| \leq \sum_m P^*(m, n - n_0)$$

where m runs through the ranges in which just one of $-f(n) < \mu(n_0, n) < f(n)$, $-f(n) - \mu_0 < \mu(n_0, n) < f(n) - \mu_0$ is satisfied. It therefore takes at most $2\mu_0$ different values. By lemma 3.21 we have for $n \geq N_3(n_0, \varepsilon)$ and all m

$$P^*(m, n - n_0) < \frac{\varepsilon}{2n_0} \leq \frac{\varepsilon}{2\mu_0}$$

It follows that $\sum_m P^*(m, n - n_0) < \tfrac{1}{2}\varepsilon$ and the lemma is proved.

3.3. If we are given that $m(n_0) = m_0$, then $P(m, n)$ $(n > n_0)$ becomes $P^*(m - m_0, n - n_0)$. It follows at once that

$$P(\,|\mu(n)| < f(n), \text{ given } m(n_0) = m_0) = P^*(-f(n) - \mu_0 < \mu(n_0, n) < f(n) - \mu_0), \quad . \; . \; . \; . \; . \quad (3.31)$$

μ_0 denoting $m_0 - \sum_1^{n_0} p_i$ as usual.

Next suppose that, instead of an exact value, we have for $m(n_0)$ the inequality $m_1 \leq m(n_0) \leq m_2$, where m_2 is the greatest integer $< \sum_1^{n_0} p + f(n_0)$ and m_1 is the least integer $> \sum_1^{n_0} p - f(n_0)$.

Then by (3.31)

$$\begin{aligned} &P(\,|\mu(n)| < f(n), \text{ given } |\mu(n_0)| < f(n_0)) \\ &= \frac{P(m_1, n_0)}{P(m_1 \leq m \leq m_2, n_0)} P^*(-f(n) - m_1 < \mu(n_0, n) < f(n) - m_1) \\ &\quad + \frac{P(m_1 + 1, n_0)}{P(m_1 \leq m \leq m_2, n_0)} P^*(-f(n) - m_1 - 1 < \mu(n_0, n) < f(n) - m_1 - 1) + \ldots \\ &\quad + \frac{P(m_2, n_0)}{P(m_1 \leq m \leq m_2, n_0)} P^*(-f(n) - m_2 < \mu(n_0, n) < f(n) - m_2). \quad . \; . \quad (3.32) \end{aligned}$$

Now $P(m, n_0) = 0$ unless $0 \leq m \leq n_0$. Remembering that $P(m_1 \leq m \leq m_2, n_0)$ is a sum of such probabilities, we see that we may suppose that m_1, m_2 on the right of (3.32) both satisfy this inequality. Lemma 3.23 is then applicable to each term, and given $\varepsilon > 0$ we have

$$|P(|\mu(n)| < f(n))\, P^*| < \varepsilon$$

for every P^* on the right of (3.32) and for all $n \geq N_2(n_0, \varepsilon)$.

From (3.32) we now have

$$P(|\mu(n)| < f(n), \text{given} |\mu(n_0)| < f(n_0)) = \frac{P(m_1, n_0)}{P(m_1 \leq m \leq m_2, n_0)} \{P(|\mu(n)| < f(n)) + \theta_m, \varepsilon\}$$
$$+ \frac{P(m_1 + 1, n_0)}{P(m_1 \leq m \leq m_2, n_0)} \{P(|\mu(n)| < f(n)) + \theta_{m_1+1}, \varepsilon\} + \dots$$
$$+ \frac{P(m_2, n_0)}{P(m_1 \leq m \leq m_2, n_0)} \cdot \quad \{P(|\mu(n)| < f(n)) + \theta_{m_2}, \varepsilon\}$$

$$P'_{n_0}(n) = P(|\mu(n)| < f(n)) + \theta_{n_0}\, \varepsilon, \quad \text{for all } n \geq N_2(n_0, \varepsilon). \quad . \quad . \quad . \quad . \quad (3.33)$$

Next consider

$$P(|\mu(n)| < f(n) \text{ for all } n \text{ in } (n', N')) = P(|\mu(n)| < f(n)\, ;\ (n', N')).$$

It is equal to

$$P(|\mu(n)| < f(n)\, ;\ (n', N' - 1) \,.\, |\mu(N')| < f(N'))$$
$$= P(|\mu(n)| < f(n)\, ;\ (n', N' - 1) \,.\, m_1 \leq \mu(N') \leq m_2)$$

(with an obvious notation)

$$= P(|\mu(n)| < f(n)\, ;\ (n', N' - 1) \,.\, \mu(N') = m_1)$$
$$+ P(|\mu(n)| < f(n)\, ;\ (n', N' - 1) \,.\, \mu(N') = m_1 + 1) + \dots$$
$$+ P(|\mu(n) < f(n)\, ;\ (n', N' - 1) \,.\, \mu(N') = m_2)$$
$$= P_{m_1} + P_{m_1+1} + \dots + P_{m_2}, \text{ say.}$$

m_1, m_2 are now functions of N', and have no connection with the previous m_1, m_2.

Then, $(n > N')$,

$$P(|\mu(n)| < f(n), \text{ given this throughout } (n', N'))$$
$$= (P_{m_1} \,.\, P'_{m_1}(n) + P_{m_1+1} P'_{m_1+1}(n) + \dots + P_{m_2} P'_{m_2}(n))/(P_{m_1} + P_{m_1+1} + \dots + P_{m_2}). \quad (3.34)$$

Now by (3.33)

$$P'_m(n) = P(|\mu(n)| < f(n)) + \theta_m\, \varepsilon, \quad \text{for all } n \geq N_2(m, \varepsilon).$$

We can therefore find $N_3 = N_3(n', N', \varepsilon)$ so that, for all $n \geq N_3(n', N', \varepsilon)$, each of the P' on the right of (3.34) differs from $P(|\mu(n)| < f(n)$ by less than ε). This being so, we have

$$P''(n) = P(|\mu(n)| < f(n), \text{ given this in } (n', N'))$$
$$= P(|\mu(n)| < f(n)) + \theta_{n', N'}\, \varepsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.35)$$

for all $n \geq N_3(n', N', \varepsilon)$.

Finally we have

$$P_1 = P(|\mu(n)| < f(n)\,;\ (n'', N''),\ \text{given this in } (n', N'))$$
$$= P(|\mu(n'')| < f(n''),\ \text{given this in } (n', N'))$$
$$. P(|\mu(n)| < f(n) \text{ in } (n'', N''),\ \text{given } |\mu(n'')| < f(n''))$$
$$= P''(n'') . P(|\mu(n)| < f(n) \text{ in } (n'', N''),\ \text{given } |\mu(n'')| < f(n''))$$

while

$$P_2 = P(|\mu(n)| < f(n)\,;\ n'', N'')$$
$$= P(|\mu(n'')| < f(n'')) . P(|\mu(n)| < f(n) \text{ in } (n''+1, N''),\ \text{given } |\mu(n'')| < f(n'')).$$

Thus

$$|P_1 - P_2| = |P''(n'') - P(|\mu(n'')| < f(n''))| . P(\mu(n) < f(n)\,;\ (n''+1, N''),\ \text{given this for } n'')$$
$$\leq |P''(n'') - P(|\mu(n'')| < f(n''))|$$
$$< \varepsilon \quad \text{for all } n'' \geq N_3(n', N', \varepsilon) \quad \ldots\ldots\ldots\ldots\ldots \quad (3.36)$$

We have now established the last statement of 3.1.

3.4. Returning to (3.12) we see that for $n'' \geq N_3(n', N', \varepsilon_0)$

$$P(n', N'') \leq P(n', N') . (P(n', N') + \varepsilon_0) < (P + \varepsilon_0)(P + 2\varepsilon_0) < P^2 + 5\varepsilon_0.$$

But

$$P(n', N'') = P + \theta_2\varepsilon_0$$

Therefore

$$P - \varepsilon_0 < P^2 + 5\varepsilon_0, \qquad P(1 - P) < 6\varepsilon_0\,;$$

ε_0 being arbitrarily small, and P being independent of ε_0, it follows that

$$P(1 - P) = 0 \qquad P = 0 \text{ or } 1.$$

3.5. We have just shown that the probability of $|\mu(n)| < f(n)$ for all n from some n_0 or other onwards is 0 or 1.

The complementary probability is the probability that $|\mu(n)| \geq f(n)$ for an infinite sequence of values of n. It follows that this also is 0 or 1.

3.6. We can apply this to KHINTCHINE's result (2.12). This asserts that, $\chi(n)$ being a certain function, the probability of $|\mu(n)| < (1 + \delta)\chi(n)$ for all n from some value onwards is 1, and the probability of $|\mu(n)| < (1 - \delta)\chi(n)$ for all n from some value onwards is 0.

But the probability of $|\mu(n)| < \chi(n)$ for all n onwards is, by what has been proved, itself 0 or 1. If it is 0 we can replace $(1 - \varepsilon)\chi(n)$ by $\chi(n)$ in KHINTCHINE's enunciation; if it is 1 we can replace $(1 + \varepsilon)\chi(n)$ by $\chi(n)$. We shall show that it is in fact always 0, and indeed that if we replace

$$\chi(n) = \sqrt{\sum_1^n 2pq \;\mathrm{ll} \sum_1^n 2pq}$$

by

$$\chi_\kappa(n) = \sqrt{\sum_1^n 2pq\left(\mathrm{ll}\sum_1^n 2pq + \tfrac{1}{2}\mathrm{l}_3\sum_1^n 2pq + \mathrm{l}_4\sum_1^n 2pq + \ldots + \mathrm{l}_\kappa\sum_1^n 2pq\right)} \qquad (\kappa \geq 4)$$

the probability that $|\mu(n)| < \chi_\kappa(n)$ for all n from some value onwards is 0. The result follows *a fortiori* for $\chi(n)$.

4.11 *Lemma.*

"If $\Psi_\kappa(n) = \sqrt{n(\mathrm{ll}\, n + \frac{1}{2}\mathrm{l}_3\, n + \mathrm{l}_4\, n + \ldots + \mathrm{l}_\kappa\, n)}$, then for all large n,

$$\mathrm{P}(\,|\,\mu(s_{n-1})\,| > \Psi_\kappa(n)\,) \quad > \quad \frac{1}{16\,\mathrm{l}\, n\ \mathrm{ll}\, n \ldots \mathrm{l}_{\kappa-1}\, n}."$$

Proof.

In (A) put $\varepsilon = \frac{1}{12}$. Let $\lambda_\kappa(n) = \sqrt{\mathrm{ll}\, n + \frac{1}{2}\mathrm{l}_3\, n + \ldots + \mathrm{l}_\kappa\, n}$, and let t_1, t_2 be defined by

$$t'_1 = -\lambda_\kappa(n), \qquad t'_2 = \lambda_\kappa(n),$$

$$m'_1 = \sum_1^{s_{n-1}} p + t'_1 \sqrt{\sum_1^{s_n} 2pq}, \qquad m'_2 = \sum_1^{s_{n-1}} p + t'_2 \sqrt{\sum_1^{s_n} 2pq},$$

$$m_1 = [m'_1], \qquad m_2 = -[-m'_2],$$

$$m_1 = \Sigma p + t_1 \sqrt{\sum_1^{s_{n-1}} 2pq}, \qquad m_2 = \Sigma p + t_2 \sqrt{\sum_1^{s_{n-1}} 2pq}\,;$$

let $\lambda_1 = m'_1 - m_1$, $\lambda_2 = m_2 - m'_2$, so that $0 \leq \lambda_1, \lambda_2 < 1$ and (as in 2.61)

$$t_1 - \frac{1}{2\sqrt{\Sigma 2pq}} = t'_1 + \frac{\mu_1}{2\sqrt{\Sigma 2pq}}, \quad t_2 + \frac{1}{2\sqrt{\Sigma 2pq}} = t'_2 + \frac{\mu^2}{2\sqrt{\Sigma 2pq}} \quad (|\,\mu_1\,|,\ |\,\mu_2\,| \leq 1).$$

From (A) we then have, since $e^{-t^2}(3t - 2t^3)$ is an odd function

$$\mathrm{P}(m'_1, m'_2) = \mathrm{P}(m_1, m_2) = \frac{1}{\sqrt{\pi}} \int_{-\lambda_\kappa(n) + \frac{\mu_1}{2\sqrt{\Sigma 2pq}}}^{\lambda_\kappa(n) + \frac{\mu_2}{2\sqrt{\Sigma 2pq}}} e^{-t^2}\, dt$$

$$+ \theta_1 \frac{1}{6\sqrt{\pi}\sqrt{\Sigma 2pq}} \int_{\lambda_\kappa(n) - \frac{\mu_1}{2\sqrt{\Sigma 2pq}}}^{\lambda_\kappa(n) + \frac{\mu_2}{2\sqrt{\Sigma 2pq}}} e^{-t^2}(3t - 2t^3)\, dt$$

$$+ \theta_2 \{(\Sigma 2pq)^{-1/2} + \tfrac{9}{8}\, \Sigma 2pq\, e^{-1/2\,(\Sigma pq)^{1/6}}\} \quad \text{for } n \geq n_0(\kappa).$$

Σ here denotes $\sum\limits_1^{s_{n-1}}$.

The last term can be written $2\theta_3/\sqrt{\Sigma 2pq}$ if n_0 is chosen large enough. And when n is large

$$|\,3t - 2t^3\,| < 2\sqrt{\pi}\, t^3,$$

so that the second term is absolutely less than

$$\frac{1}{3\sqrt{\Sigma 2pq}} \int_{\lambda_\kappa(n) - \frac{1}{2\sqrt{\Sigma 2pq}}}^{\lambda_\kappa(n) + \frac{1}{2\sqrt{\Sigma 2pq}}} e^{-t^2} t^3\, dt < \frac{1}{3\sqrt{\Sigma 2pq}} \int_{\lambda_\kappa(n)}^{\lambda_\kappa(n) + \frac{1}{\sqrt{\Sigma 2pq}}} e^{-u^2+1} \left(u - \frac{1}{2\sqrt{\Sigma 2pq}}\right)^3 du,$$

(n large)

$$< \frac{1}{\Sigma 2pq} \frac{(\lambda_\kappa(n))^3}{\mathrm{l}\, n \sqrt{\mathrm{ll}\, n} \ldots \mathrm{l}_\kappa\, n} < \frac{1}{\Sigma 2pq}.$$

Thus

$$1 - \mathrm{P}(m'_1, m'_2) > \frac{2}{\sqrt{\pi}} \int_{\lambda_\kappa(n)+\frac{1}{2\sqrt{\Sigma 2pq}}}^{\infty} e^{-t^2}\,dt - \frac{2}{\sqrt{\Sigma 2pq}} - \frac{1}{\Sigma 2pq}$$

$$> \frac{2}{\sqrt{\pi}} \int_{\lambda_\kappa(n)}^{\infty} e^{-t^2}\,dt - \frac{4}{\sqrt{\Sigma 2pq}}.$$

Now

$$\int_{t_0}^{\infty} e^{-t^2}\,dt > \frac{e^{-t_0^2}}{8t_0} \quad (t_0 \geq \tfrac{1}{2}) \quad \text{(see 2.6).}$$

Therefore for $n \geq n_0(\varepsilon)$

$$1 - \mathrm{P}(m'_1, m'_2) > \frac{1}{4\sqrt{\pi}} \cdot \frac{1}{\mathrm{l}\,n \sqrt{\mathrm{ll}\,n}\,\mathrm{l}_3\,n \ldots \mathrm{l}_{\kappa-1}\,n} \cdot \frac{1}{\sqrt{\mathrm{ll}\,n + \ldots}} - \frac{4}{\sqrt{\Sigma 2pq}}$$

$$> \frac{1}{4\sqrt{\pi}\,\mathrm{l}\,n\,\mathrm{ll}\,n \ldots \mathrm{l}_{\kappa-1}\,n} - \frac{4}{\sqrt{n-1}} > \frac{1}{16\,\mathrm{l}\,n\,\mathrm{ll}\,n \ldots \mathrm{l}_{\kappa-1}\,n}.$$

4.12 *Lemma.*

"If $\chi_\kappa(n) = \sqrt{\sum_1^n 2pq\,(\mathrm{ll} \sum_1^n 2pq + \tfrac{1}{2}\mathrm{l}_3 \sum_1^n 2pq + \ldots + \mathrm{l}_\kappa \sum_1^n 2pq)}$ then for all $n \geq n_1(\kappa)$

$$\mathrm{P}(|\mu(s_n)| > \chi_\kappa(s_n)) > 1/17\,\mathrm{l}\,n\,\mathrm{ll}\,n \ldots \mathrm{l}_{\kappa-1}\,n."$$

Proof.

$\chi_\kappa(s_{n-1}) < \psi_\kappa(n)$; hence

$$\mathrm{P}(|\mu(s_{n-1})| > \chi_\kappa(s_{n-1})) \geq \mathrm{P}(|\mu(s_{n-1})| > \psi_\kappa(n))$$

$$> 1/16\,\mathrm{l}\,n\,\mathrm{ll}\,n \ldots \mathrm{l}_{\kappa-1}\,n$$

$$> 1/17\,\mathrm{l}(n-1)\,\mathrm{ll}(n-1) \ldots \mathrm{l}_{\kappa-1}(n-1) \quad (n \text{ large}),$$

i.e.,

$$\mathrm{P}(|\mu(s_n)| > \chi_\kappa(s_n)) > 1/17\,\mathrm{l}\,n\,\mathrm{ll}\,n \ldots \mathrm{l}_{\kappa-1}\,n.$$

4.2. Now let

$$\mathrm{E}(r, n) = \sum_{r+1}^{n} 2pq \qquad (r < n)$$

and let

$$n_i = [\mathrm{A}^{i\,\mathrm{ll}\,i}], \quad (i > e^e)$$

$$= \mathrm{A}^i, \quad (i < e^e) \qquad \ldots\ldots\ldots\ldots \quad (4.21)$$

A being a (large) positive integer.

Let $s_{n_1}, s_{n_2}, \ldots$ be successively the least positive integers such that

$$[\mathrm{E}(0, s_{n_1})] = n_1, \ [\mathrm{E}(s_{n_1}, s_{n_2})] = n_2, \ldots \ [\mathrm{E}(s_{n_{i-1}}, s_{n_i})] = n_i, \ldots \quad \ldots\ldots \quad (4.22)$$

(This definition of s_{n_i} is evidently different from that previously adopted.)

Let $\mu(s_{n_{i-1}}, s_{n_i}) = \mu(s_{n_i}) - \mu(s_{n_{i-1}})$ as usual. Then by lemma 4.12, if A be chosen $\geq n_1(\kappa)$ we have

$$\mathrm{P}(|\mu(s_{n_{i-1}}, s_{n_i})| > \sqrt{n_i\,(\mathrm{ll}\,n_i + \tfrac{1}{2}\mathrm{l}_3\,n_i + \ldots + \mathrm{l}_\kappa\,n_i)})$$

$$> 1/17\,\mathrm{l}(n_i+1)\,\mathrm{ll}(n_i+1) \ldots \mathrm{l}_{\kappa-1}(n_i+1)$$

$$> 1/18\,\mathrm{l}\,n_i\,\mathrm{ll}\,n_i \ldots l_{\kappa-1}\,n_i \quad \text{if A be chosen} \geq n_2(\kappa)\ (\geq n_1(\kappa))$$

$$> \mathrm{K}(\mathrm{A})/i \log i \log_2 i \ldots \log_{\kappa-2} i \quad \text{for } i = 1, 2, 3, \ldots \infty,$$

or, denoting this probability by π_i,

$$\pi_i > \mathrm{K}/i \log i \log_2 i \ldots \log_{\kappa-2} i.$$

4.3. Now let P_i denote the probability that the i inequalities

$$\left\{\begin{array}{ll} |\mu(s_{n_i})| > \chi_{\kappa-1}(n_i) & \quad (4.31) \\ |\mu(s_{n_1})| \leq \chi_{\kappa-1}(n_1), \quad |\mu(s_{n_2})| \leq \chi_{\kappa-1}(n_2), \quad \ldots \quad |\mu(s_{n_{i-1}})| \leq \chi_{\kappa-1}(n_{i-1}) & \quad (4.32) \end{array}\right.$$

are simultaneously true. Then $\sum\limits_1 \mathrm{P}_i$ is the probability that (4.31) is satisfied for at least one $i \leq t$.

Consider the trials from the $(s_{n_{i-i}}+1)$th to the s_{n_i}th ; the probability π_i that

$$|\mu(s_{n_{i-1}}, s_i)| > \sqrt{n_i(\mathrm{ll}\, n_i + \tfrac{1}{2}\mathrm{l}_3\, n_i + \ldots + \mathrm{l}_\kappa\, n_i)} \quad \ldots \quad (4.33)$$

satisfies, by (4.2), the inequality

$$\pi_i > \mathrm{K}/i \log i \log_2 i \ldots \log_{\kappa-2} i. \quad \ldots \quad (4.34)$$

Suppose that for a particular i, (4.33) and (4.32) are both satisfied. Then we shall show that, provided A be chosen large enough, (4.31) is also satisfied.

We have in fact

$$|\mu(s_{n_i})| \geq |\mu(s_{n_{i-1}}, s_{n_i})| - |\mu(s_{n_{i-1}})|$$
$$> \sqrt{n_i(\mathrm{ll}\, n_i + \tfrac{1}{2}\mathrm{l}_3\, n_i + \ldots + \mathrm{l}_\kappa\, n_i)} - \sqrt{n_{i-1}(\mathrm{ll}_{n_{i-1}} + \ldots + \mathrm{l}_{\kappa-1}\, n_{i-1})}$$

which for sufficiently large A

$$> \sqrt{n_i(\mathrm{ll}\, n_i + \tfrac{1}{2}\mathrm{l}_3\, n_i + \ldots + \mathrm{l}_{\kappa-1}\, n_i)}\left(\sqrt{1+\frac{\mathrm{l}_\kappa\, n_i}{2\,\mathrm{ll}\, n_i}} - \sqrt{\frac{n_{i-1}}{n_i}}\right)$$
$$> \chi_{\kappa-1}(n_i)\left(1+\frac{\mathrm{l}_\kappa\, n_i}{3\,\mathrm{ll}\, n_i} - \frac{2}{\mathrm{A}^{1/2\,\mathrm{ll}\, i}}\right) \qquad (i \geq e^e + 1)$$

since $n_{i-1} \leq \mathrm{A}^{(i-1)\,\mathrm{ll}\,(i-1)} < \mathrm{A}^{(i-1)\,\mathrm{ll}\, i}$, $n_i > \frac{1}{4}\mathrm{A}^{i\,\mathrm{ll}\, i}$

$$> \chi_{\kappa-1}(n_i) \text{ for } i \geq i_0, \text{ since } n_i \leq \mathrm{A}^{i\,\mathrm{ll}\, i},\ \mathrm{ll}\, n_i < \mathrm{l}\, i + \mathrm{l}_3\, i + \mathrm{ll}\,\mathrm{A} < 2\,\mathrm{l}\, i.$$

Thus if $i \geq i_0$, (4.31) follows from (4.32), (4.33). It follows that for $i \geq i_0$

$$\mathrm{P}_i \geq \pi_i\left(1 - \sum_{1}^{i-1} \mathrm{P}_j\right),$$

and hence, as before, that

$$\sum_1^t \mathrm{P}_i \to 1 \text{ as } t \to \infty.$$

In other words we can assert with probability 1 that

$$|\mu(s_{n_i})| < \chi_{\kappa-1}(n_i) \quad \ldots \quad (4.36)$$

for an infinite sequence of values of i.

When such a sequence exists, we can select from it a subsequence $i_0, i_1, i_2, \ldots$ such that

$$\chi_{\kappa-1}(n_{i_r}) > \chi_{\kappa-2}(n_{i_r} + n_{i_{r-1}} + \ldots + n_{i_1} + n_{i_0} + r). \quad \ldots \quad (4.37)$$

For suppose $i_0, i_1 \ldots i_{r-1}$ chosen. Let

$$a = n_{i_{r-1}} + n_{i_{r-2}} + \ldots + n_{i_0} + r;$$

then we have to make

$$\chi_{\kappa-1}(n_{i_r}) > \chi_{\kappa-2}(n_{i_r} + a).$$

Now to make

$$\chi_{\kappa-1}(n) > \chi_{\kappa-2}(n + a)$$

we have to make

$$n(\mathrm{ll}\, n + \tfrac{1}{2}\mathrm{l}_3\, n + \ldots + \mathrm{l}_{\kappa-1}\, n) > (n+a)(\mathrm{ll}(n+a) + \ldots + \mathrm{l}_{\kappa-2}(n+a)).$$

When $a > 0$,

$$\mathrm{ll}(n+a) = \mathrm{ll}\, n + a/(n+\theta a)\,\mathrm{l}(n+\theta a) < \mathrm{ll}\, n + a/n$$

$$\tfrac{1}{2}\mathrm{l}_3(n+a) < \tfrac{1}{2}\mathrm{l}_3\, n + a/n \ldots \mathrm{l}_{\kappa-1}(n+a) < \mathrm{l}_{\kappa-1}\, n + a/n;$$

thus we need only make

$$n(\mathrm{ll}\, n + \tfrac{1}{2}\mathrm{l}_3\, n + \ldots + \mathrm{l}_{\kappa-1}\, n) > (n+a)\left(\mathrm{ll}\, n + \ldots + \mathrm{l}_{\kappa-2}\, n + \frac{(\kappa-1)\,a}{n}\right)$$

$$n\,\mathrm{l}_{\kappa-1}\, n > (\kappa-1)\,a + a\left(\mathrm{ll}\, n + \ldots + \mathrm{l}_{\kappa-2}\, n + \frac{(\kappa-1)\,a}{n}\right),$$

and this is satisfied for $n \geq n_0(a, \kappa)$.

We can therefore choose $i_0, i_1, \ldots$ successively, so that (4.37) holds. We then have, from (4.36) and (4.22)

$$|\mu(s_n)| > \chi_{\kappa-2}(n)$$

for an infinite sequence of values of n, or

$$|\mu(n)| > \chi_{\kappa-2}\left(\sum_1^n 2pq\right)$$

for an infinite sequence of values of n. Thus finally:

"We can assert with probability 1 that

$$|\mu(n)| > \sqrt{\sum_1^n 2pq\left(\mathrm{l}_2 \sum_1^n 2pq + \tfrac{1}{2}\mathrm{l}_3 \sum_1^n 2pq + \ldots + \mathrm{l}_\kappa \sum_1^n 2pq\right)} \quad . \quad . \quad (4.38)$$

for an infinite sequence of values of n."